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Theory of the Landau Domain Structure for Thin Samples

Jorge Marug

J. M. Simonin* and **A. López**

Centro Atómico Bariloche, Instituto Balseiro,†‡ *S.C. de Bariloche, Argentina*

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We present a theory of the Landau domain structure for Type I superconductors which is applicable in the limit of vanishing penetration depth and for slab widths smaller than the coherence length. Results for the perpendicular critical field compare quite satisfactorily with experiments on Al and Cd.

1. INTRODUCTION

Since the pioneering work of Landau[1] on the intermediate state of type I superconductors, the original theory has been extended by several authors to improve it in various ways. Krempasky and Farrell[2] discussed an extension that included the effect of an inclined field, using a simplified shape of the normal–superconducting (NS) boundary. Miyazaki[3] considered a general boundary shape and minimized the free energy with respect to it.

Landau explicitly considered the limit of very thick samples, compared to both λ and the period of the repeating structure. Miyazaki relaxed the condition on the sample thickness, by considering the mutual influence of both surfaces. Although it is not clearly stated there, the calculation of Ref. 3 is strictly valid only in the limit of vanishing penetration depth λ. In this limit, which is the same we consider here, there is no magnetic field penetration into the superconducting regions, so that magnetism and superconductivity influence each other only through the geometry of the respective domains.[4]

Our interest in this problem arose from the measurements of the transverse critical field on very thin Al and Cd slabs by Maloney *et al.*[5] and de la Cruz *et al.*[6] The thicknesses ranged between 1 and 200 μm for the Al samples and between 4 and 1000 μm for Cd (κ values 0.013 and 0.012). These results cannot be successfully accounted for by the simple relation

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†Universidad Nacional de Cuyo.

‡Comisión Nacional de Energía Atómica.

between critical field and thickness that follows from the Landau model,[4]

$$H_{C\perp} = H_c[1 - 1.88(\Delta/d)^{1/2}]^{1/2} \tag{1}$$

where d is the slab thickness and Δ is the surface energy parameter.

The point has been raised[7] that, since a decrease in sample thickness causes the size of the superconducting region to decrease, a situation could be reached in which the value of the order parameter in a superconducting domain could be reduced relative to the maximum it could attain in a bulk sample, the latter being $|\psi_\infty|^2 = -\alpha/\beta = mc^2/8\pi e^2\lambda^2$, where α, β, and λ are the parameters of the Ginzburg–Landau theory.

This situation calls for a careful theoretical analysis of the intermediate state of thin samples. The models of Refs. 1 and 2 assume a constant magnetic field equal to H_c at the center of the slab and along the whole of the normal region, from one NS boundary to the next. For very thin slabs this assumption is certainly not correct. The nonuniform compression of the field lines caused by the presence of the superconducting domain will persist all the way across the material.

This point has been dealt with in Ref. 3. However, in this paper as well as in Refs. 1 and 2, the superconducting free energy is written as a sum of a volume term and a surface term, which follow by considering that superconductivity has attained its maximum bulk value.

In the present paper we have tried to define a mathematically consistent model that is also physically sound. Our calculation of the free energy starts from the Ginzburg–Landau expression and we show under what conditions it can be written as a sum of a volume plus a surface term.

We find that the transition is of first order for all thicknesses. As the field approaches the perpendicular critical field $H_{c\perp}$, the period increases indefinitely and the size of the superconducting domains decreases to a minimum value of order 4ξ, at which point the transition to the normal state takes place. Within the laminar model, change to a regime where second-order transitions take place[9] is only possible if one allows for a finite penetration depth. This would progressively weaken the average order parameter so it can go continuously to zero.

In Section 2 we describe the model and state the fundamental equations used. Section 3 contains a discussion of the superconducting and magnetic free energies. Section 4 gives an analysis of the equilibrium conditions and of the phase transition. A comparison with experiments is also given there.

2. THE MODEL

We consider a slab of thickness d placed perpendicular to the external field H_0 and assume that when cooled below its critical temperature the sample breaks up into equally spaced superconducting and normal strips of width $2a_s$ and $2a_n$, respectively. These laminae are assumed to extend indefinitely in the z direction. We want to set up a model valid for $\lambda \ll \xi$ and $d \leq \xi$.

The following assumptions are consistent with the above range of parameters:

(i) There is no variation of the order parameter f within a superconducting domain in the direction of the field (y axis). This is based on the fact that variations of f can occur only over distances of the order of ξ.

(ii) There is no magnetic field penetration into the superconducting regions. This assumption is certainly correct in the limiting case $\lambda \ll \xi$.

Assumptions (i) and (ii) mean that the superconducting and magnetic energies can be evaluated separately.

Within a superconducting domain the order parameter can vary in the x direction, normal to the NS interface, according to the GL equation[4]:

$$\xi^2\, d^2f/dx^2 = f(f^2 - 1) \tag{2}$$

We choose the origin of the x axis at the center of the superconducting region, so f varies from a maximum value f_0 at $x = 0$ to zero at the NS boundary $f(\pm a_s) = 0$.

The magnetic field H satisfies the free field equations within the normal domains and in the adjacent free space above and below the sample.

The values of a_s and a_n should be chosen to minimize the total free energy difference, which is the sum of a superconducting GL free energy ΔF_S, plus the deformation energy of the field due to the presence of the superconducting domains, ΔF_M.

The superconducting free energy difference ΔF_S for a half-period and per unit length in the z direction, is given in the GL theory by

$$\Delta F_S = \frac{H_c^2 d}{4\pi} \int_0^{a_s} dx \left[-f^2 + \frac{f^4}{2} + \xi^2 \left(\frac{df}{dx} \right)^2 \right] \tag{3}$$

The magnetic free energy relative to the normal state is given by[1]

$$\Delta F_M = -\tfrac{1}{2} \mathbf{M} \cdot \mathbf{H}_0 \tag{4}$$

where **M** is the magnetization due to the superconducting surface currents and $\mathbf{H}_0$ is the applied field.

3. THE FREE ENERGY

3.1. Superconductive Free Energy

The solution to Eq. (2) can be obtained in the following way. Define the parameters φ and k according to

$$f = f_0 \sin \varphi \tag{5a}$$

$$f_0^2 = 2k^2/(1+k^2) \tag{5b}$$

The relation between x and the pair (φ, k) is given by

$$a_s - x = \xi(1+k^2)^{1/2}F(k, \varphi) \tag{6}$$

where $F(k, \varphi)$ is the elliptic integral of the first kind.[8] The half-width a_s is related to k, and thus to f_0, through

$$a_s = \xi(1+k^2)^{1/2}F(k, \pi/2) \tag{7}$$

The free energy difference ΔF_S can be cast into the form

$$\Delta F_S = \frac{2}{3}\frac{H_c^2}{4\pi}\frac{\xi d}{(1+k^2)^{3/2}}[2(1+k^2)E(k) - (2+k^2)K(k)] \tag{8}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively.[8] From (5b) and (7) we see that when f_0 varies between one and zero, a_s sweeps the range from infinity to $\pi\xi/2$ monotonically.

Expression (8) for the free energy of one-half the superconducting domain tends, in the limit $a_s \gg \xi$, to

$$\Delta F_S = (H_c^2/8\pi)\, d(-a_s + \Delta) \tag{9}$$

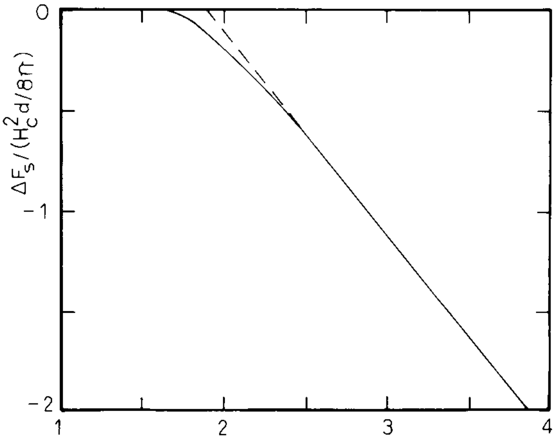

Fig. 1. A comparison between the exact form of the LG free energy difference and the asymptotic expression which is a sum of condensation energy plus surface energy. The solid line is the exact expression; the dashed line is the asymptotic form.

where

$$\Delta = (4\sqrt{2}/3)\xi$$

As is apparent from Eq. (9), the free energy in this limit is given as a sum of the terms that are usually considered when dealing with the problem of the intermediate state.[1-3] The first term is the condensation energy and the second one is the surface energy, with the parameter Δ which corresponds to the limit $\kappa \ll 1$. Figure 1 shows the variation of ΔF_S as a function of a_s as given by Eqs. (8) and (9). As we shall see, the superconducting-to-normal transition takes place for $a_s \approx 4\xi$. It follows from Fig. 1 that the asymptotic expression (9) is quite a good approximation for the whole range of interest.

3.2. The Magnetic Free Energy

From the fact that the magnetic field is curl-free in the region of interest, it can be derived from a scalar potential which satisfies the Laplace equation. The field configuration can be obtained similarly to what was done in Refs. 1 and 3, by means of a conformal mapping. Figure 2 shows the complex t and w planes. The region in the $t = y + ix$ plane consists of an infinite strip of height equal to a half period $a = a_s + a_n$. Because of our assumption that there is no y dependence of the order parameter and no field penetration into the superconducting regions, the cross sections of the domains are right-angled, as shown in Fig. 2.

The conformal mapping to the w plane, where the region of interest is transformed into the upper half-plane, is determined by the differential relation

$$\frac{dt}{dw} = \frac{aA}{w^2 - (\pi/2)^2}\left(\frac{w^2 - \theta_1^2}{w^2 - \theta_2^2}\right)^{1/2} \tag{10}$$

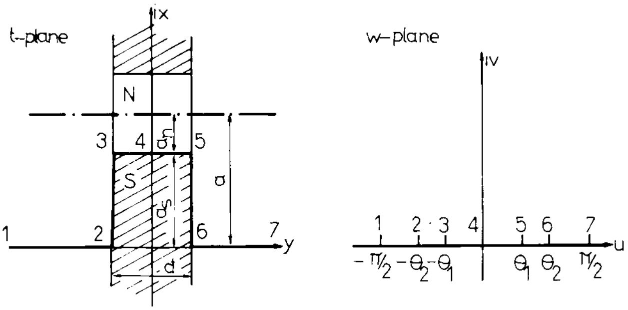

Fig. 2. Complex t and w planes which are connected through conformal mapping. The hatched region in the t plane is half the superconducting domain. A strip of height a in the t plane is transformed into the upper half-plane.

The magnetic field is given by

$$H_y - iH_x = \frac{H_0}{A}\left(\frac{w^2-\theta_2^2}{w^2-\theta_1^2}\right)^{1/2} \tag{11}$$

where

$$A = \left(\frac{(\pi/2)^2-\theta_2^2}{(\pi/2)^2-\theta_1^2}\right)^{1/2} \tag{12}$$

The field along the line 3-5 of Fig. 2 varies according to

$$H_y = \frac{H_0}{A}\left(\frac{\theta_2^2-u^2}{\theta_1^2-u^2}\right)^{1/2} \tag{13}$$

and diverges on reaching points 3 and 5, i.e., at the right angles. This is a consequence of our assumption of a vanishing penetration length. A finite, however small, value for λ would result in a variation of f along y with a corresponding rounding off of the contours[4] of Fig. 2.

The relation between geometrical parameters in both complex planes is given through

$$\frac{d}{2a} + i\frac{a_s}{a} = \frac{\theta_1 A}{(\pi/2)^2}\int_0^{\pi/2} \frac{[1-(\theta_2/\theta_1)^2 \sin^2\varphi]^{1/2}}{1-(2\theta_2/\pi)^2 \sin^2\varphi}\, d\varphi \tag{14}$$

The magnetic free energy can be written as

$$AF_m = \frac{H_0^2}{8\pi}\frac{a^2\theta_1 A}{(\pi/2)^3} \times \int_\alpha^{\pi/2} \frac{[(\theta_2/\theta_1)^2 \sin^2\varphi - 1]^{1/2}}{1-(2\theta_2/\pi)^2 \sin^2\varphi} \operatorname{arctanh}\left(\frac{2}{\pi}\theta_2 \sin\varphi\right) d\varphi \tag{15}$$

where $\alpha = \sin^{-1}(\theta_1/\theta_2)$.

4. EQUILIBRIUM CONDITIONS

The equilibrium configuration for a given external field results from minimizing the total free energy difference per unit length with respect to the parameters θ_1 and θ_2.

We have studied numerically the function

$$\frac{\Delta F}{a} = \frac{\Delta F_m + \Delta F_S}{a}$$

varying θ_1 and θ_2, and we found that the minimum always occurs for those θ_1 and θ_2 such that the field at point 4 (Fig. 2) equals H_c. This gives the following relation between θ_1 and θ_2:

$$H_4 = H_c = \frac{H_0}{A}\frac{\theta_2}{\theta_1} \tag{16}$$

Within the limitations of our simplified model this is the condition for phase equilibrium at the boundary. That the field increases when one moves toward point 3 or 5 is not in contradiction with the fact that this line is a phase boundary, because of our assumption of zero penetration length.

Relation (15) allows us to search for the equilibrium configuration by varying only one parameter. The equations are still involved and one must deal with them numerically. However, the condition for the critical field H can be cast in a rather simple form by noting that at the transition field the parameter a_n diverges. This implies that for $H = H_c$, $\theta_2 = 0$. In this way we obtain the following relation between the thickness of the foil d and the perpendicular critical field $h_\perp = H_{c\perp}/H_c$:

$$d = \frac{2\,\Delta I_1}{I_2 - \frac{1}{8}\pi(h_\perp^2/I_1)(1 - h_\perp^2)} \tag{17}$$

Here I_1 and I_2 are the real and imaginary parts of the following integral:

$$I_1 + iI_2 = \int_0^{\pi/2} (h_\perp^2 - \sin^2\varphi)^{1/2}\, d\varphi$$

and $\Delta = (4\sqrt{2}/3)\xi$.

The half-width of the superconducting region at the transition field is given by

$$a_{\text{sc}} = \tfrac{1}{2} d I_2 / I_1 \tag{18}$$

Numerical evaluation of (17) shows that a_{sc} is almost independent of d for the whole range of applicability of the present model and its value is $\sim 4\xi$.

Figure 3 shows the results for the perpendicular critical field $H_{c\perp}/H_c$ as function of d/ξ. It is seen that there is reasonably good agreement with the experimental results for both Al and Cd for d/ξ between 0.1 and 10. For thicker samples our assumption of a rigid order parameter across the sample is surely not applicable. In this region the Landau model becomes more appropriate. In the limit of very thin samples it has been predicted[9] that the laminar structure could be energetically less favorable than a spot-like structure in which there are superconducting islands arranged in an orderly manner in a normal background. For even thinner samples the Abrikosov vortex state should be preferred and the transition becomes second order.

The phase transition can be described by means of two order parameters. A short-range order parameter is defined by considering the

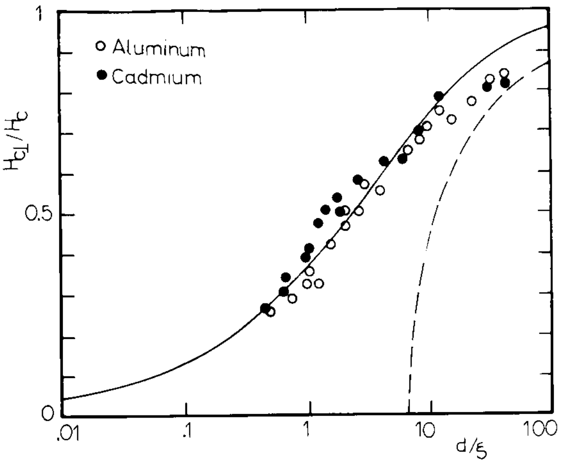

Fig. 3. Perpendicular critical field $H_{c\perp}/H_c$ as a function of sample thickness d/ξ. The solid line is the result of the present calculation. Open circles are experimental points for Al from Ref. 5. Solid circles are measurements for Cd from Ref. 6. The dashed line is the result of the Landau model for $\Delta = (4\sqrt{2}/3)\xi$.

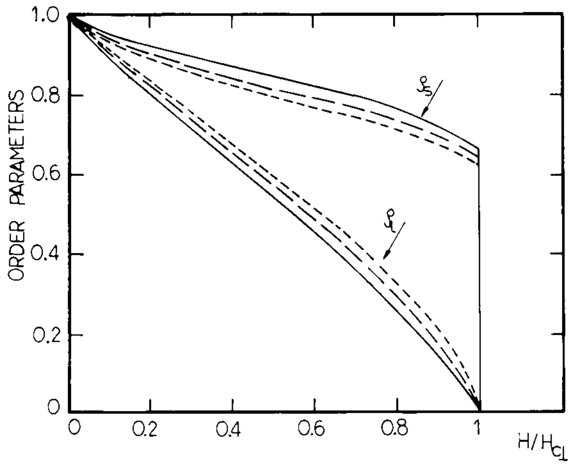

Fig. 4. Short-range and long-range order parameters, as defined in the text, for different slab thicknesses. Short-dashed, long-dashed and solid lines are for $d/\xi = 0.1$, 1, and 5, respectively.

average value of the squared Ginzburg–Landau order parameter in one superconducting domain,

$$\rho_s = \frac{1}{a_s}\int_0^{a_s} dx\, f^2(x) = 1 - \frac{\sqrt{2}\xi}{a_s}$$

A long-range or average order parameter is given by the average of f^2 over one period of the repeating structure

$$\rho_L = \frac{1}{a}\int_0^{a} dx\, f^2(x) = \rho_s \frac{a_s}{a}$$

Figure 4 shows a plot of these two quantities as functions of magnetic field. The order of the transition should be defined from the behavior of ρ_s, the short-range order parameter.

In Fig. 5 we have plotted the size a_s of the superconducting region and the total period a. We see that the size of the superconducting regions decreases up to a minimum value of the order of 4ξ at the transition. This implies, as we said above, that the asymptotic form of the superconducting free energy, Eq. (8), can be used as a very good approximation for the whole range of values of magnetic field up to the critical field.

The theoretical results could be used to fit data for materials with higher κ values by adjusting the surface energy parameter Δ.

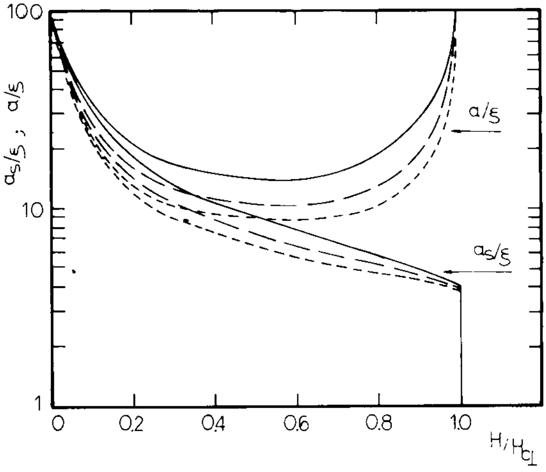

Fig. 5. Size of superconducting domain and half-period for different sample thicknesses. The lines are as in Fig. 4.

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