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NUMERICAL EXPERIMENTS ON THE DETERMINATION OF UNSTEADY STATE TEMPERATURE DISTRIBUTION IN DOMAINS OF COMPLICATED BOUNDARY SHAPE

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This paper compares analytical and finite element results for an unsteady heat-conduction problem in simply and doubly connected plates of regular polygonal shape. A numerical solution is obtained by means of the powerful finite element method and the results are shown to agree with an approximate conformal mapping-variational technique previously developed by the first author and coworkers.

1. Introduction

The popular finite element method permits virtually any problem of mechanics to be tackled in a unified mathematical fashion suitable for solution in a digital computer.

Applications of the method to the solution of a wide variety of problems ranging from the design of prostheses to the dynamic analysis of nuclear, naval and aerospace structures, are known nowadays. Typical problems include cylindrical or prismatic configurations of odd-shaped cross sections, e.g. a graphite brick of a gas-cooled nuclear reactor, solid propellant rocket motors, and waveguides commonly used in microwave engineering.

Most of the studies performed to evaluate the relative accuracy of the finite element method deal with comparisons between alternative formulations of the method or between exact analytical results and finite element values obtained in the case of rather simple geometries.

Ref. [1] presents a completely independent comparison for diffusion-type problems in cases where the geometry of the domain is complex. However, the problems attacked in that study are of the steady state type.

The goal of the present paper is the comparison of finite element results and analytical solutions for unsteady temperature fields in domains of regular polygonal shape (simply and doubly connected regions). The analytical approach was presented in refs. [2] and [3], and only a brief treatment will be presented here. It is important to point out that only the simply connected square plate admits an exact solution. *

2. The complex variable-variational approach

If the thermal conductivity is constant and no heat is generated within the solid, two-dimensional heat flow in an isotropic solid is governed by:

$$\alpha \nabla^2 T = \partial T / \partial t , \qquad (1)$$

where α is the thermal diffusivity and ∇^2 the two-dimensional Laplacian operator.

* Available in standard textbooks.

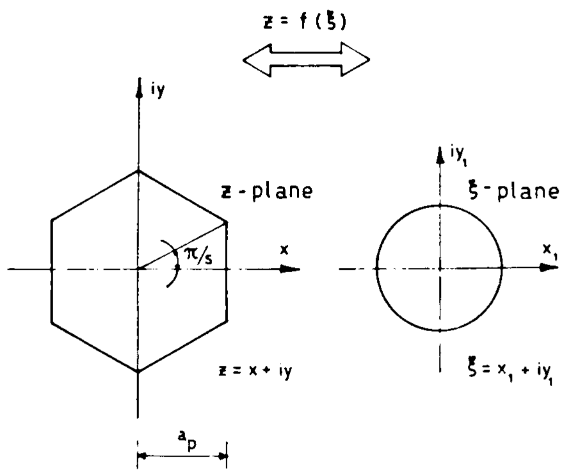

Fig. 1. Conformal mapping of a regular polygonal shape.

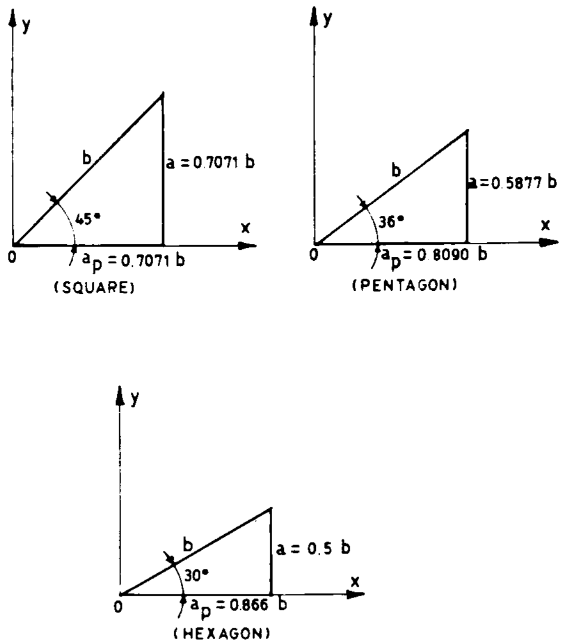

Fig. 2. Subdomains of the regular polygonal plates.

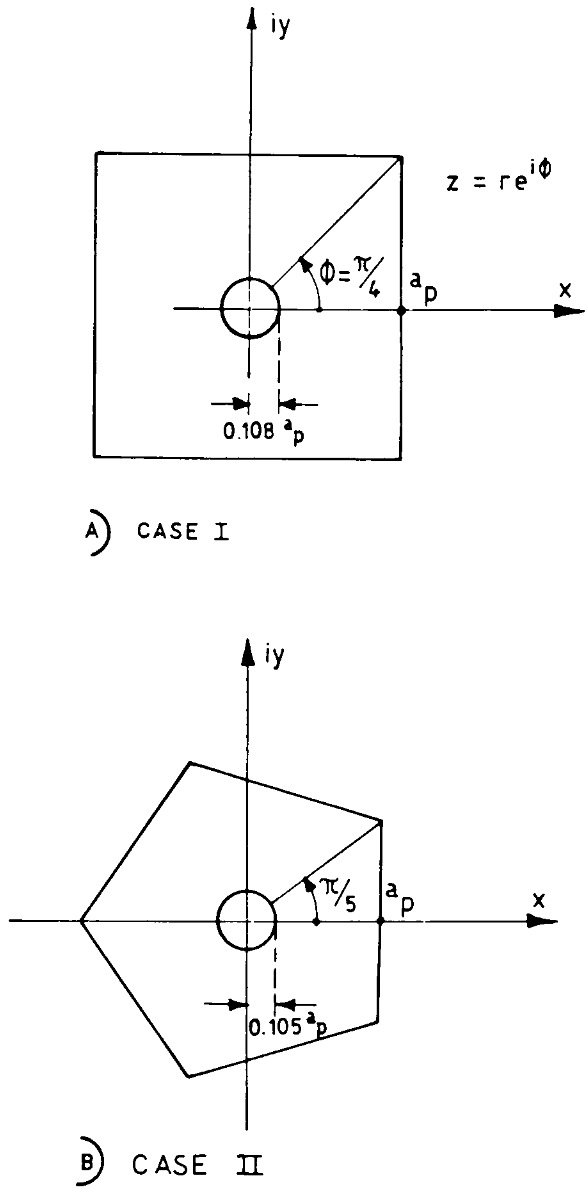

Fig. 3. Doubly connected regions (a_p = apothem of the regular polygon).

Let the functional relations

$$T[L(x, y) = 0, t] = 0 \,, \tag{2a}$$

$$T[x, y, t]_{t=0} = T_0 \,, \tag{2b}$$

where $L(x, y) = 0$ is the functional relation which defines the boundary configuration, represent the boundary and initial conditions, respectively, for the thermo-mechanical system under study.

If

$$z = x + iy \tag{3}$$

and

$$z = f(\xi) \tag{4}$$

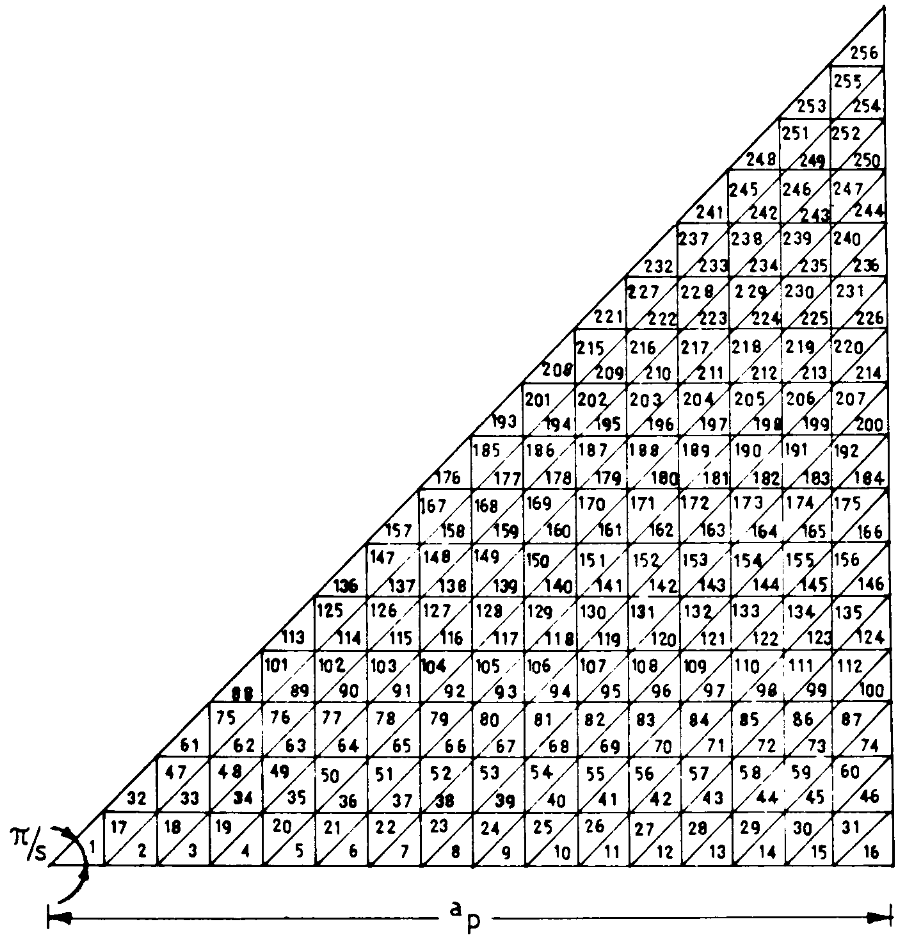

Fig. 4. Element distribution.

represents the analytic function which maps the given domain in the z-plane onto a unit circle in the ξ-plane (fig. 1), it can be shown [2] that the temperature field in the ξ-plane can be described by the functional relation

$$T(\xi, \bar{\xi}) \simeq \sum_{m=1}^{M} A_{0m} J_0[\beta_{0m}(\xi \cdot \bar{\xi})^{1/2}]\, e^{-\alpha\gamma_{0m}t}, \qquad (5)$$

where J_0 is the Bessel function of the first kind and order zero; the β_{0m}'s are the roots of $J_0(x)$ and the γ_{0m}'s are the separation constants obtained when one substitutes $T = T_1(x, y)\,\theta(t)$ in eq. (1).

The expansion coefficients A_{0m} are given by the Fourier–Bessel expansion

$$A_{0m} = \frac{2T_0}{\beta_{0m} J_1(\beta_{0m})}. \qquad (6)$$

Ref. [3] deals with an approximate analytical formulation applicable to doubly connected domains when the governing differential system is also defined by (1) and (2). Clearly, instead of eq. (2a) one has now

$$T[L_i(x, y) = 0, t] = 0\,, \qquad (i = 1, 2)\,, \qquad (7)$$

where the subscript 1 denotes the inner boundary and 2 the outer.

Numerical results are presented in ref. [3] for square and pentagonal plates with concentric circular holes.

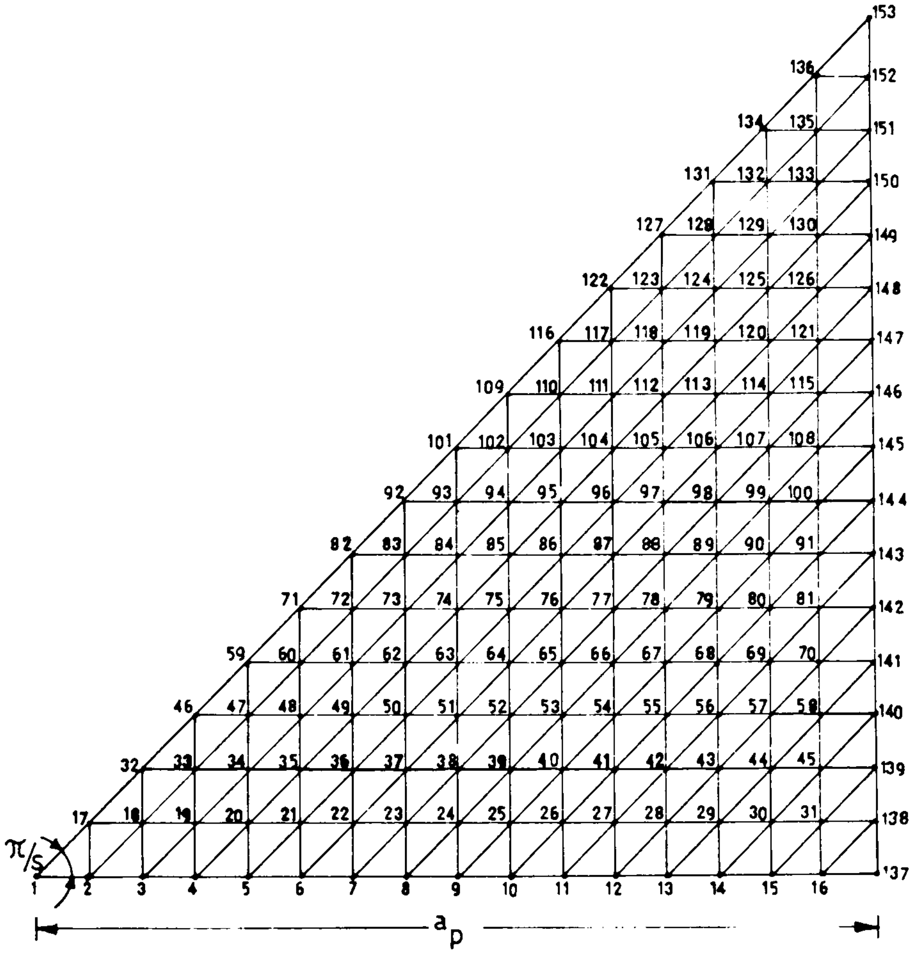

Fig. 5. Nodal pattern.

3. The finite element solution

The results were obtained using a finite element code. * The domain is subdivided into triangular elements and a linear variation of the temperature field is assumed inside the element.

It was decided, in view of the symmetry, to consider the subdomains defined by

$$0 \leqslant \phi \leqslant \pi/s \tag{8}$$

where s is the order of the polygon with the conditions (see fig. 2)

$$T = 0\,, \qquad \text{on } a \tag{9a}$$

and

$$\frac{\partial T}{\partial n} = 0\,, \qquad \text{on } a_p \text{ and } b\,, \tag{9b}$$

where n denotes the outer normal to the subdomain.

In the case of doubly connected plates (fig. 3) the condition (9a) was also applied to the circular portion of the boundary limited by the relation (8).

Figs. 4 and 5 show the element distribution in the case of simply connected shapes, used in the present analysis † (256 elements and 153 nodal points).

Calculations were performed for (a) square, penta-

* Developed at Centro Atómico Bariloche, Comisión Nacional de Energía Atómica.

† The element distribution was similar when analyzing the doubly connected plates. The number of elements and nodes was the same.

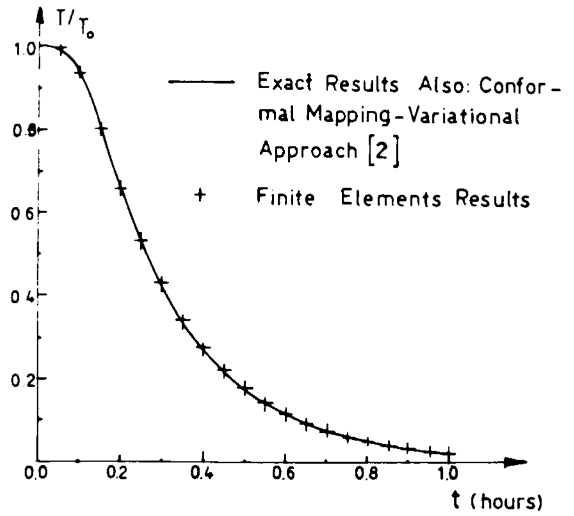

Fig. 6. Variation of the dimensionless temperature parameter T/T_0 at the center of a square plate.

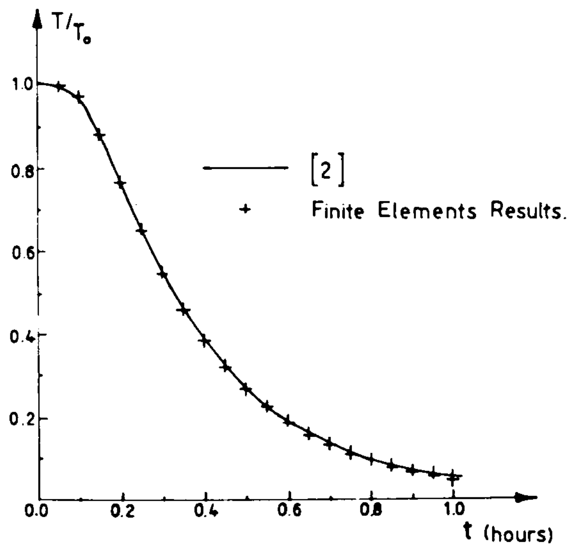

Fig. 7. Variation of the dimensionless temperature parameter T/T_0 at the center of a pentagonal plate.

gonal and hexagonal simply connected plates, and (b) square and pentagonal plates with concentric circular holes.

Table 1
Values of T/T_0 at the center of a square plate: comparison of results

t (hr)	Exact solution	Conformal mapping-variational approach [2]	Finite elements
0.1	0.928	0.925	0.933
0.2	0.654	0.649	0.659
0.3	0.426	0.422	0.428
0.4	0.274	0.271	0.274
0.5	0.176	0.174	0.175
0.6	0.113	0.112	0.112
0.7	0.0724	0.0716	0.0718
0.8	0.0464	0.0459	0.0459
0.9	0.0298	0.0295	0.0294
1.0	0.0191	0.0189	0.0188

4. Comparison of results and conclusions

Table 1 shows a comparison of results obtained [+] by means of (a) exact solution, (b) complex variable-variational formulation [2], and (c) the finite element method, in the case of a simply connected square domain.

The agreement is indeed quite good [the mean square error of (b) and (c) with respect to the exact formulation is less than 1%].

Fig. 6 depicts graphically the results of table 1. Figs. 7 and 8 display comparisons of results for pentagonal and hexagonal shapes. The agreement can be again considered as quite satisfactory (one observes a more marked difference in the case of a hexagonal domain for $t > 0.6$ hr).

Figs. 9–12 depict several comparisons between results obtained in ref. [3] and values calculated using the finite elements technique (results are plotted in terms of the dimensionless parameters r/a_p and $\tau = \alpha t/a_p^2$ in order to compare with results published in ref. [3]). It may be concluded that the agreement is, in general, quite reasonable.

No claim of originality is made in the present paper. On the other hand the present study probably

[+] In the case of the simply connected shapes: $b = 1$ ft and $\alpha = 0.45$ ft^2/hr in order to compare with calculations performed in ref. [2].

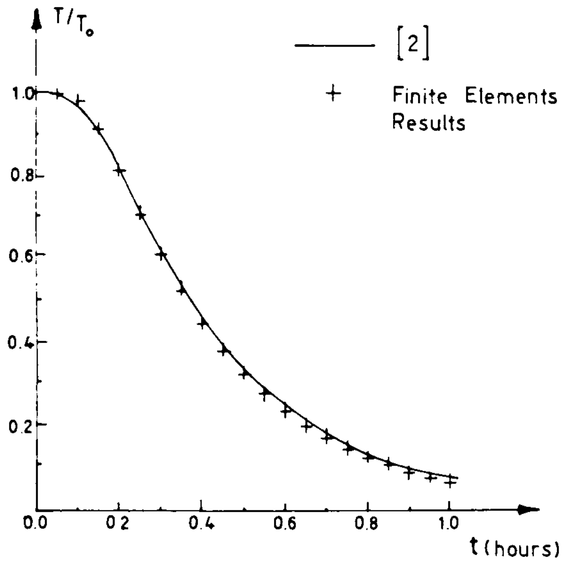

Fig. 8. Variation of the dimensionless temperature parameter T/T_0 at the center of a hexagonal plate.

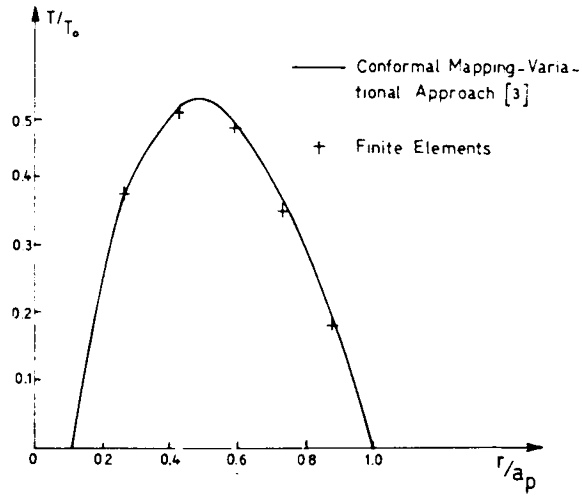

Fig. 9. Square shape with circular perforation: variation of T/T_0 as a function of r/a_p for $\phi = 0$; $\tau = 0.10$.

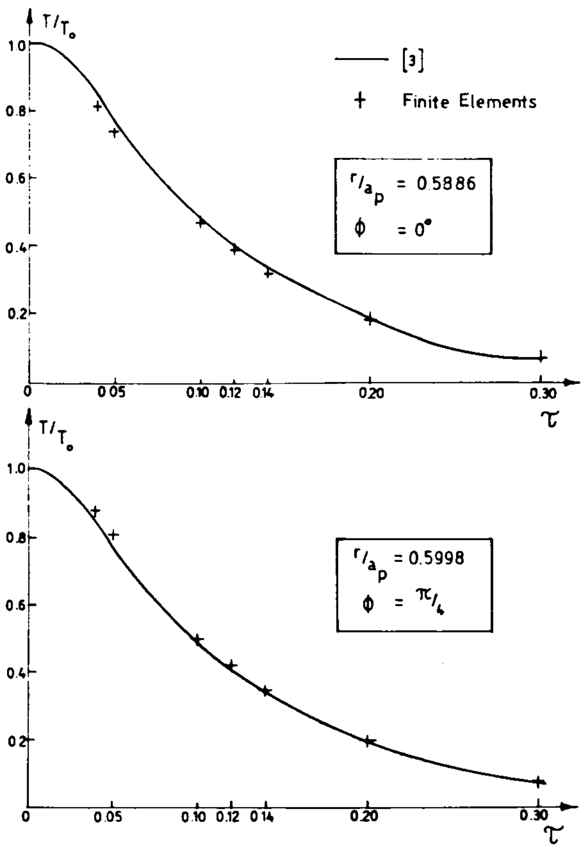

Fig. 10. Square shape with circular perforation: variation of T/T_0 for two different points of the domain.

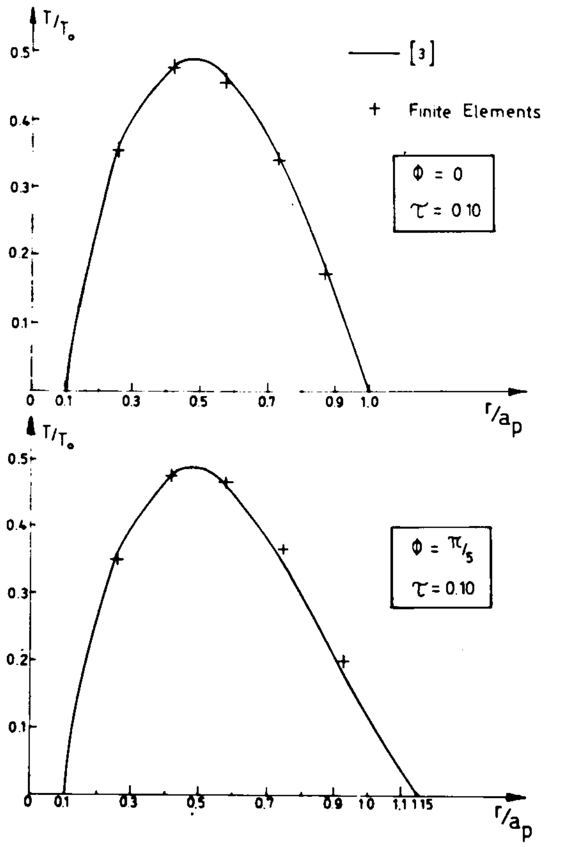

Fig. 11. Pentagonal shape with circular perforation: variation of T/T_0 as a function of r/a_p for a particular value of τ.

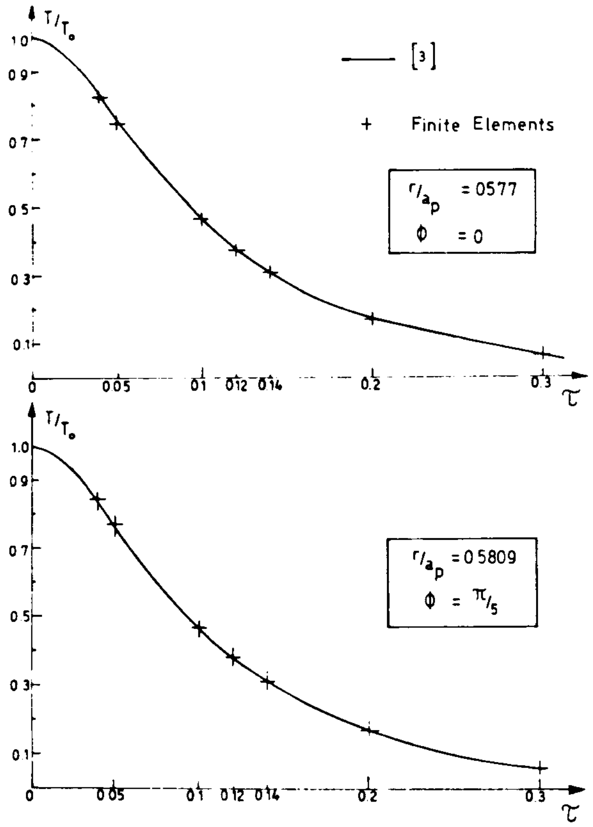

Fig. 12. Pentagonal shape with circular perforation: variation of T/T_0 for two different points of the domain.

constitutes one of the first 'experimental' evaluations of the relative precision of the finite element method in the case of unsteady diffusion phenomena in complicated boundary shapes.

References

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[2] P.A.A. Laura, J.A. Reyes and R.E. Rossi, Nucl. Eng. Des. 31 (1974) 379.

[3] P.A.A. Laura and R. Ercoli, Nucl. Eng. Des. 23 (1972) 1.