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UN METODO DE SEGUIMIENTO DE LA INTERFASE PARA PROBLEMAS UNIDIMENSIONALES DE FRONTERA MOVIL

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A FRONT TRACKING METHOD FOR ONE-DIMENSIONAL MOVING BOUNDARY PROBLEMS*

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ABSTRACT

We introduce a front tracking method for the numerical solution of one-dimensional Stefan problems. It consists in the formulation of the Stefan problem as an ordinary differential initial-value problem for the moving boundary coupled with a parabolic partial differential equation for the distribution of temperatures. We present a variable time step procedure in which the initial-value problem is solved with a predictor-corrector scheme; in the corrector step the function evaluation is done, iteratively, through an implicit time discretization of the parabolic equation. Numerical results for one-dimensional,one-phase Stefan problems with straight and curved moving boundary trajectories are presented. For these cases the front tracking method presented gives greatly improved results.

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UN METODO DE SEGUIMIENTO DE LA INTERFASE

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RESUMEN

Se introduce un método numérico de seguimiento de un frente móvil para la solución de problemas de Stefan unidimensionales. El método consiste en la formulación del problema de Stefan como un problema de ecuaciones diferenciales ordinarias de valores iniciales para el seguimiento de la interfase acoplado con una ecuación a derivadas parciales de tipo parabólico para el problema de difusión. Se presenta un procedimiento de cálculo de paso variable de tiempo en el que el problema de valores iniciales se resuelve con un método predictor-corrector; en el paso corrector la evaluación funcional se efectúa, iterativamente, a través de una discretización implícita de la ecuación parabólica. Se presentan resultados numéricos para problemas de Stefan unidimensionales de una sola fase y con trayectorias de la interfase recta y curva. Para este tipo de problemas el método de seguimiento de la interfase presenta ventajas considerables.

1. INTRODUCTION

A moving boundary problem or Stefan problem is a nonlinear initial boundary value problem with a moving boundary whose position is unknown and has to be determined as part of the solution.

Among the various methods in current use for studying the Stefan problem finite difference and finite element methods are widely used, and have proven to be the most general. They can be classified into two main categories: the methods which explicitly track the interface or front thus solving the differential problem in a time varying domain, and those which use a fixed domain with the help of a weak or generalized formulation. By a suitable coordinate transformation it is possible to obtain methods that can be also included in the last category. An enlightening discussion of these methods can be found in the works of Crank /3/ and Furzeland /5/.

Finite difference one-dimensional moving boundary tracking methods, hereafter called front tracking methods, have been classified by Gupta /6/ as fixed and variable grid methods. In the fixed grid method the space-time domain is subdivided into a finite number of equidistributed cells and the trajectory of the front not necessarily coincides with the cell nodes. In the variable grid method the space-time domain is subdivided into a finite number of rectangular cells with only one side equidistributed, the other side being subdivided in such a way that the trajectory of the front coincides with the cell nodes. Examples of fixed grid methods can be found, for instance, in the works of Crank /2/, Meyer /10/ and Furzeland /5/. Douglas and Gallie /4/ were the first to introduce a variable grid method; they subdivide the space direction into an equally distributed mesh length and determine the time step in such a way that the trajectory of the front advances one space cell per time step. For the solution of the nonlinear system they introduce an iteration procedure by which, at every time step, an arbitrary initial guess for the time step is chosen and the parabolic equation is then solved using a strongly implicit time discretization scheme; for the next iteration the new time step is corrected using a quadrature of the integral form of the moving boundary condition. Stability and uniform convergence of the method in a suitable restricted domain were also proven. However, Gupta and Kumar /6/ showed that the previous method did not converge under more rigorous conditions, and presented a new variable time step method which is closely related to the Douglas and Gallie method. It differs in the way in which the new time step is corrected: Gupta and Kumar use a first order finite difference approximation of the differential form of the moving boundary condition.

In the present work we propose a new front tracking method which shares some of the ideas discussed above. It consists in the formulation of the one-dimensional Stefan problem as an ordinary differential initial-value problem governing the moving boundary coupled with a parabolic partial differential equation describing the diffusion process. This slightly different formulation, that grants equal importance to the moving boundary and to the diffusion process leads naturally to a consistent difference approximation. We introduce a variable time step procedure in which the initial-value problem is approximated with a second order predictor-corrector scheme; in the corrector step the function evaluation is performed, iteratively, through the use of an implicit time discretization of the parabolic partial differential equation. As it will be shown, this front tracking method yields a simple, robust, fast and accurate numerical procedure. We present numerical results for two test cases involving straight line and curved line interface trajectories and a comparison with other authors' results.

2. FORMULATION OF THE STEFAN PROBLEM

We reformulate the Stefan problem using the test case presented by Douglas and Gallie /4/. One-dimensional , one-phase Stefan problems consist in finding the space-time curve $S(t)$ along which the front moves satisfying the initial-value problem

$$\dot{S} = \frac{dS}{dt} = -\frac{\partial u}{\partial x}, \qquad x = S(t), \; t > 0 \tag{2.1}$$

$$S(t) = 0, \qquad t = 0 \tag{2.2}$$

and the value of the function $u(x,t)$ in the region $0 \leq x \leq S(t)$ satisfying the following parabolic partial differential equation for the function $u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}, \qquad 0 \leq x \leq S(t) \qquad t > 0 \tag{2.3}$$

with initial and boundary conditions

$$u = 0, \qquad 0 \leq x \leq S(t), \quad t = 0 \tag{2.4}$$

$$\frac{\partial u}{\partial x} = -1, \qquad x = 0, \quad t > 0 \tag{2.5}$$

$$u = 0, \qquad x = S(t), \quad t > 0 \tag{2.6}$$

An alternative formulation is to replace equations (2.1) and (2.2) by its integral form

$$S(t) = t - \int_0^{S(t)} u \, dx \tag{2.7}$$

It can easily be shown (see for instance Hirsch and Tadjbakhsh /1/) that the Stefan problem is a nonlinear problem. Performing the coordinate transformation $z = x - S(t)$ system (2.1)-(2.6) becomes

$$\dot{S} = \frac{dS}{dt} = - \frac{\partial u}{\partial z} \quad , \quad z = 0 \quad , \quad t > 0 \tag{2.8}$$

$$s(t) = 0 \; , \tag{2.9}$$

$$\frac{\partial u}{\partial t} = \dot{S} \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} \quad , \quad - S(t) \leq z \leq 0 \; , \; t > 0 \tag{2.10}$$

$$u = 0, \quad - S(t) \leq z \leq 0 \; , \; t = 0 \tag{2.11}$$

$$\frac{\partial u}{\partial z} = - 1 \; , \quad z = - S(t) \; , \quad t > 0 \tag{2.12}$$

$$u = 0 \; , \quad z = 0 \; , \quad t > 0 \tag{2.13}$$

The convective term appearing in equation (2.10) has a coefficient which depends on the unknown $u(z,t)$.

3. THE NUMERICAL PROBLEM

This section discusses numerical approximations for the Stefan problem (2.1)-(2.6) using front tracking methods. The space-time domain $0 \leq x \leq S(t)$ is discretized with a rectangular grid which distribution is discussed below. Space and time increments are indicated by h and k, respectively. As usual the value of the numerical solution for $x = i\,h$ and $t = n\,k$ (i and n integers) is given by $u(x,t) = u(ih,nk) = u_i^n$. Assuming that the space-time curve of the front is a parabola, the following strategies for the discretization of the domain can be envisaged: a) to make a uniform mesh distribution in space with h fixed for all times and a variable mesh distribution in time with k chosen in such a way that the front is always kept at the rightmost node at any time, see Fig. 3.1; b) to fix the value of the time step (not necessarily constant for all times) and to search for the position of the front using a fixed space step for all times, except for the last node which is variable, see Fig. 3.2. The advantages and inconveniences of these methods will be later discussed. Now we introduce the finite difference approximation utilized. Assuming that the front position and the function $u(x,t)$ are known at the n time level, as shown in Fig. 3.3, the problem is to obtain the front position and the function $u(x,t)$ at the next time level. To this end the initial-value problem (2.1)-(2.2) describing the front movement is approximated with a predictor-corrector scheme of the form

$$\frac{S^{*n+1} - S^n}{k} = \alpha \; F(S(t),u_x)^n \tag{3.1}$$

$$\frac{S^{n+1} - S^n}{k} = \beta \; F(S^*(t),u_x^*)^{n+1} + (1-\beta) \; F(S(t),u_x) \tag{3.2}$$

where α and β are weighting factors lying in the interval $0 \leq \alpha , \beta \leq 1$, and $F(S(t),u_x)^n$ and $F(S(t^*),u_x^*)^{n+1}$ represent the right hand side of equation (2.1) evaluated at the time level t_n and t_{n+1}, respectively. The intermediate function evaluation is obtained via the solution of equation (2.3) with an implicit time discretization scheme of the form

$$\frac{u_i^{*n+1} - u_i^n}{k} = \theta \, \delta u_i^{*n+1} + (1-\theta) \quad \delta u_i^n \qquad (3.3)$$

where δ is the standard three point centered finite difference operator and θ is a weighting factor lying in the interval $0 \le \theta \le 1$; for $\theta = 0$, 1/2 and 1, explicit, Crank-Nicolson and strongly implicit schemes, respectively, are obtained. Thus, to advance the solution from the n to the n+1 time level, equations (3.1) to (3.3) are solved iteratively and convergence is achieved if in two successive iterations the values of the unknowns differ in less than a prescribed precision (usually 10^{-6}).

Returning to the discussion of the two methods described at the beginning of this section we note that in the first method, which is usually called a variable time step method, $t = t(S)$, therefore the initial-value problem (1.1)-(1.2) is formulated interchanging the role of the dependent and independent variables. In the second method, $S = S(t)$ and the initial-value problem remains in its original form. When the trajectory of the front has a slope $\dot{S}$ near unity any method will do; however, if $\dot{S}$ tends to zero (for instance when a stationary state is sought) clearly the first method cannot be used, since at some point of the calculations the independent variable will fall outside of the domain and the numerical method will fail to converge. Similarly, the second method will fail when $\dot{S}$ tends to infinite. In this work we discuss the first method; the second method is presented in Marshall and Smith /9/. Further details about the approximation used for the right hand side of the predictor-corrector scheme (3.1)-(3.2) as well as for the difference scheme (3.3) are given in the appendix.

It can be shown that the finite difference system (3.1)-(3.2) and (3.3) is consistent with the differential problem (1.1)-(1.6). A linear stability analysis shows that a necessary condition for the numerical stability of the corrector scheme (3.2) is

$$k \le \frac{2}{1-2\beta}\,\frac{1}{F_S} \qquad \text{for } 0 \le \beta < 1/2 \qquad (3.4)$$

$$\text{no restriction} \qquad \text{for } \beta \ge 1/2 \qquad (3.5)$$

where F_S is the maximum value, between the time levels n and n+1, of the derivative of $F(S(t),u_x)$ with respect to S. This stability condition is valid only if F_S remains positive but this is guaranteed by the physics of the problem. In the same fashion a linear stability analysis shows that a necessary condition for the numerical stability of the implicit time discretization (3.3) is

$$k \le \frac{h^2}{2\,(1-2\theta)} \qquad \text{for } 0 \le \theta < 1/2 \qquad (3.6)$$

$$\text{no restriction} \qquad \text{for } \theta \ge 1/2 \qquad (3.7)$$

The stability conditions (3.4) to (3.7) which were derived for the second method are valid also for the first method, except that in equation (3.4) the restriction is in the space step rather than in the time step and a strict inequality holds. Numerical evidence has shown that the stability analysis presented remains valid for the nonlinear case.

4. NUMERICAL RESULTS

We present numerical results obtained with the front tracking method applied to straight and curved front trajectories. The accuracy of the results obtained by the front tracking method is checked against exact and approximate analytical solutions; the efficiency of the method is compared with that of other numerical techniques.

The straight line front trajectory

This problem which is taken from Furzeland /5/ and is due to Hoffman /7/ is described by equations (1.1) to (1.6) with the following modification for the boundary condition (2.5)

$$\frac{\partial u}{\partial x} = - e^{t} \quad , \quad x = 0 \ , \quad t > 0 \tag{4.1}$$

The exact solution of this problem is

$$u(x,t) = e^{t-x} - 1 \tag{4.2}$$

and the trajectory of the front in the x-t plane is the straight line

$$S(t) = t \tag{4.3}$$

Figure 4.1 illustrates the exact solution in the x-t plane. The computations were done on an IBM 370/158 (under CMS-VM) with a Fortran compiler, in single precision, using the front tracking method. We found that the most accurate results were obtained using a three point lagrangian interpolation for the approximation of the front slope (all the calculations use this procedure). Table I shows the exact and computed position of the front , for different values of the mesh and of the numerical parameters. These results are shown to converge to 10^{-6} in seven iterations for h = 0.1, two iterations for h = 0.025 and one iteration for meshes not coarser than h = 0.01. This last rather remarkable result is mainly due to the presence of a predictor scheme which is consistent with the differential problem. The number of inner iterations is primarily a function of the precision sought. For fixed precision, the number of iterations decreases as the mesh is refined, for the simple reason that an initial guess based on the previous time level solution is closer to the final solution when the time step is smaller. Obviously, for fixed precision and mesh size the number of inner iterations is a function of the numerical scheme used.

It was also found that for fine meshes the weighting coefficients β and θ do not affect appreciably the number of inner iterations but do affect the accuracy of the results. In this connection it would be expected the most accurate results to occur for values of β and θ near 0.5; however, the presence of oscillations, typical of Crank-Nicolson type schemes, makes that the most accurate results are attained for values of β and θ near 0.9 (Table I shows that the error -- measured as the difference between the numerical and exact solutions -- decreases in time to a minimum, changes its sign and increases thereafter). For values of β and θ equal to one the error increases monotonically (not shown here).

For comparative purposes we reproduce in Table II some of the results obtained by Furzeland /5/ for the same problem using the Landau coordinate transformation, first with an implicit time discretization method (first and second columns), and then with the method of lines in space (third column). Regarding his first method, which is directly comparable to the front tracking method, Furzeland reports having reached a precision of 10^{-6} with two and three inner iterations. The front tracking method needs only one iteration and its results are more accurate, even more so considering that the word length used is shorter. The results of Furzeland's second method, the accuracy of which is nothing less than remarkable, can be compared with the front tracking results, for the same grid, in relation to accuracy and simplicity (leaving aside the computing time). In relation to accuracy, the superiority of Furzeland's results is only apparent considering that we have used single precision and a shorter word length (the NAG ODE package requires double precision arithmetic). The implementation of the method of lines in space is, indeed, very attractive: ease of programming and use of a robust ODE integrator package. So is the tracking method which consists in the solution of only one ODE involving the inversion of a tridiagonal system; it can be improved further using an automatic ODE solver rather than a predictor-corrector scheme. Moreover, the front tracking method has none of the inconveniences of the other methods discussed here. The latter need a coordinate transformation, and, in addition, the global accuracy of the method of lines depends on the spatial discretization. As this is refined, the ODE system increasingly stiffens. It is concluded in the light of the evidence presented that the front tracking method appears to be superior.

The curved line front trajectory

This problem which is taken from Douglas and Gallie /4/ is described by equations (1.1) to (1.6). An approximate analytic solution of this problem was given by Gupta and Kumar /6/ using Goodman's integral method

$$u(x,t) = A(x-S) + B(x-S)^2 \tag{4.4}$$

where $S = S(t)$ and

$$A = \frac{1 - \mathrm{sqrt}(1+4S)}{2S}$$

$$B = \frac{1+2S-\mathrm{sqrt}\ (1+4S)}{4S^2}$$

The trajectory of the interfaoe is given by

$$t = \frac{S}{6}\ (5+S+\mathrm{sqrt}(1+4S)) \tag{4.5}$$

In Fig. 4.2 it is illustrated the approximate analytic solution.

Table III presents the results obtained with the front tracking method for different values of the mesh and the approximate analytic position of the front. We found that the optimum value of β and θ is one. The results converged to 10^{-6} in eight iterations for h=0.1, four iterations for h = 0.025 and two iterations for h = 0.01. The number of inner iterations is higher as compared with the previous test problem because of the curved trajectory of the front. Moving along the rows of Table III (from coarser to finer meshes) it is possible to see that the results tend asymptotically to a fixed value, which is a practical proof of the convergence of the method. In passing we note that the approximate analytic solution gives good results only for values of S(t) that are not too far from the origin. Table VI shows the results obtained by Gupta and Kumar /6/ for the same problem using an implicit time discretization and a first order approximation for the treatment of the front slope, together with the results obtained with the front traoking method. The results of Gupta and Kumar converge to a precision of 0.5 percent apparently after several iterations (not specified by the authors); the tracking method results converge to the same precision in one iteration. Comparison with the results of the approximate analytic solution (for values near the origin), that can be considered the most accurate, reveals that of the two finite difference methods, front tracking is the most accurate and the fastest. This can be attributed to the use of a higher approximation for the treatment of the front slope, and of a predictor scheme which is consistent with the differential problem.

5. CONCLUSIONS

A new front tracking method for the numerical treatment of one-dimensional moving boundary problems has been presented. It consists in the formulation of the Stefan problem as an ordinary differential equation for the moving boundary coupled to a parabolic equation for the diffusion process. A variable time step procedure has been introduced in which the ordinary differential equation is solved with a predictor-corrector scheme and the parabolic equation with an implicit time discretization. Numerical experiments with straight and curved front trajectories and comparison with other author's results evidenced that the front tracking method is an efficient, simple and accurate numerical procedure which produces greatly improved results.

6. REFERENCES

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7. APPENDIX

Here we discuss in some detail the implementation of the front tracking method. The right hand side of equations (3.1) and (3.2) involves the approximation of the slope of the front as given by equation (2.1). This can be done, for instance, using linear or quadratic Lagrange polynomials:

$$\left(\frac{\partial u}{\partial x}\right)_j = (u_j - u_{j-1})/h \tag{A.1}$$

$$\left(\frac{\partial u}{\partial x}\right)_j = (3\, u_j - 4\, u_{j-1} + u_{j-2}) \;/\; (2\, h) \tag{A.2}$$

Equations (A.1) and (A.2) are two-point and three-point backward approximations of the first derivative of $u(x,t)$ at the j node. Using a fictitious node it is possible to obtain a centred approximation:

$$\left(\frac{\partial u}{\partial x}\right)_j = (\, u_{j+1} - u_{j-1}) \;/\; (2\, h) \tag{A.3}$$

The fictitious centred approximation is locally the most accurate due to the nature of its truncation error; however, numerical evidence shows that with a quadratic Lagrange polynomial a better global accuracy is attained.

The implicit time discretization (3.3) of the parabolic equation (2.3) can be written (dropping the star superindex for convenience)

$$\frac{u_i^{n+1} - u_i^{n}}{k} = \frac{\theta}{h^2}\left(u_{i-1} - 2u_i + u_{i+1}\right)^{n+1} + \frac{(1-\theta)}{h^2}\left(u_{i-1} - 2u_i + u_{i+1}\right)^{n} \tag{A.4}$$

which is valid for values of i lying in the interval $1 \le i \le j$. Rearranging equation (A.4) we obtain

$$-\,\theta r\, u_{i-1}^{n+1} + (1+2\theta r)\, u_i^{n+1} - \theta r\, u_{i+1}^{n+1} = D_i^{n} \tag{A.5}$$

where

$$D_i^n = (1-\theta) r \, u_{i-1}^n + \{1-2(1-\theta)r\} \, u_i^n + (1-\theta) r \, u_{i+1}^n$$

The boundary conditions are incorporated as follows. For i = 1 a centred difference approximation of equation (2.5) yields

$$\frac{u_2 - u_0}{2h} = -1 \quad , \qquad (A.6)$$

replacing in equation (A.5) for $i = 1$ and rearranging

$$(1+2\theta r) \, u_1^{n+1} - 2\theta r \, u_2^{n+1} = D_1^n \qquad (A.7)$$

where

$$D_1^n = \{1-2(1-\theta)r\} \, u_1^n + 2(1-\theta) r \, u_2^n - 2rh$$

The right boundary oondition (2.6) at i = j yields

$$-\theta r \, u_{j-1}^{n+1} + (1+2\theta r) \, u_j^{n+1} = D_j^n \qquad (A.8)$$

where D_j^n is the same expression as in (A.5) but for $i = j$. The value of u_{j+1}^n in equation (A.8) is obtained by an extrapolation procedure which uses equations (A.1), (A.2) or (A.3). Finally the tridiagonal system of order j becomes

$$\begin{bmatrix} 1+2\theta r & -2\theta r & \\ -\theta r & 1+2\theta r & -\theta r \\ & & 1+2\theta r \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_i^{n+1} \\ u_j^{n+1} \end{bmatrix} = \begin{bmatrix} D_1^n \\ D_i^n \\ D_j^n \end{bmatrix} \qquad (A.9)$$

This algebraic system can be easily inverted using an upper-lower triangular decomposition (see for instance Isaacson and Keller /8/ or Richtmyer and Morton /11/).

It is worthwhile to examine the starting algorithm. Here we propose the following procedure. Let us assume the situation depicted in Fig. A.1. We wish to compute the numerical solution at the time level t_{n+1}. To this end we write the following difference approximation at node i = 1, for the initial-value problem (2.1) and for the parabolic equation (2.3) and its boundary conditions:

$$\frac{S^{n+1} - S^{n}}{k} = - \frac{u_2^{n+1} - u_1^{n+1}}{h} \qquad \text{(A.10)}$$

$$- r\, u_0^{n+1} + (1 + 2r)\, u_1^{n+1} - r\, u_2^{n+1} = u_1^{n} \qquad \text{(A.11)}$$

$$\frac{u_2^{n+1} - u_0^{n+1}}{2h} = -1 \qquad \text{(A.12)}$$

$$u_2^{n+1} = u_1^{n} = 0,$$

where we have used a first order approximation for the front slope and a strongly implicit time discretization for the parabolic equation. Recalling that in the variable time step method k is the unknown ($t_{n+1} - t_n = k$, time being the dependent variable), and $h = S^{n+1} - S^{n}$ is given, we obtain a nonlinear system of three equations with three unknowns: k, u_0^{n+1} and u_1^{n+1}. The nonlinearity of system (A.10)-(A.11)-(A.12) can easily be established replacing the value of $r = k/h^2$ obtained from equation (A.10) into equation (A.11). However, for this particular case, an explicit solution is available, by solving the quadratic equation in u_1^{n+1} (here called u for convenience)

$$u^2 + p\,u + q = 0 \qquad \text{(A.13)}$$

where p = 2 and q = - 2 h.

TABLE I Position of the straight line front trajectory as calculated with the front tracking method.

space	time (computed)			time (exact)
	$h = 0.10$ $\beta = 0.88$ $\theta = 0.88$	$h = 0.025$ $\beta = 0.89$ $\theta = 0.89$	$h = 0.010$ $\beta = 0.90$ $\theta = 0.90$	
0.0	0.0000	0.0000	0.0000	0.0
0.1	0.1000	0.1001	0.1001	0.1
0.2	0.2000	0.2003	0.2002	0.2
0.3	0.3008	0.3005	0.3002	0.3
0.4	0.4013	0.4005	0.4002	0.4
0.5	0.5016	0.5005	0.5002	0.5
0.6	0.6017	0.6004	0.6002	0.6
0.7	0.7015	0.7003	0.7002	0.7
0.8	0.8012	0.8001	0.8001	0.8
0.9	0.9008	0.8999	0.9000	0.9
1.0	1.0002	0.9997	1.0000	1.0
CPU IBM-370/158 (seconds)	0.10	0.23	0.67	

TABLE II Position of the straight line front trajectory as calculated by Furzeland /5/.

time	space (computed)			space (exact)
	implicit time discretization		method of lines	
	$h = 0.01$ $\theta = 1$	$h = 0.01$ $\theta = 0.5$	$h = 0.025$	
0.1	0.1000	0.1003	0.1003	0.1
0.2	0.2001	0.2003	0.2003	0.2
0.5	0.5003	0.5003	0.5003	0.5
0.9	0.9006	0.9003	0.9003	0.9
1.0	1.0007	1.0003	1.0003	1.0
CPU ICL 1906A (seconds)	2	2	12	

TABLE III Position of the curved line front trajectory as calculated with the front tracking method

space	time (computed) h = 0.1 β = 1.0 θ = 1.0	h = 0.025 β = 1.0 θ = 1.0	h = 0.01 β = 1.0 θ = 1.0	time (integral method)
0.0	0.0000	0.0000	0.0000	0.0000
0.3	0.3450	0.3421	0.3403	0.3392
0.6	0.7582	0.7491	0.7450	0.7444
0.8	1.0646	1.0507	1.0452	1.0568
1.0	1.3927	1.3745	1.3675	1.3727
1.5	2.3004	2.2724	2.2611	2.2865
2.0	3.3225	3.2862	3.2691	3.3333
2.5	4.4500	4.4062	4.3833	4.5069
3.0	5.6762	5.6257	5.5944	5.8028
CPU IBM 370/158 (sec)	0.52	3.89	11.3	

TABLE IV Position of the curved line front trajectory as calculated by Gupta and Kumar /6/ and with the front tracking method.

space	time (computed)				time (integral method)
	Gupta and Kumar /6/		tracking method		
	h = 0.1 θ = 1.0	h = 0.01 θ = 1.0	h = 0.1 θ = 1.0 β = 1.0	h = 0.01 θ = 1.0 β = 1.0	
0.2	0.2091	0.2172	0.2230	0.2190	0.2181
0.6	0.7186	0.7406	0.7616	0.7458	0.7444
1.0	1.3285	1.3604	1.3937	1.3688	1.3727
1.6	2.3944	2.4413	2.4934	2.4559	2.4854
2.0	3.1993	3.2522	3.3176	3.2725	3.3333
2.6	4.5340	4.5916	4.6792	4.6211	4.7564
3.0	5.5035	5.5599	5.6696	5.5955	5.8028

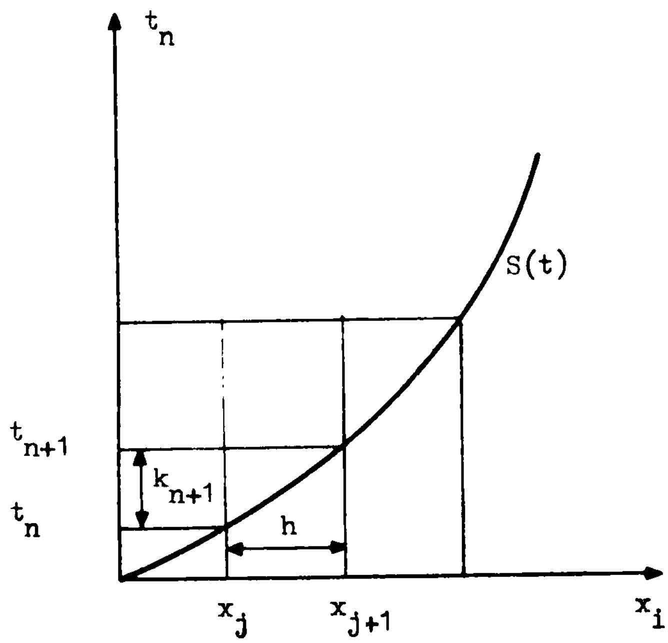

Fig. 3.1 Uniform mesh distribution in space and variable mesh distribution in time.

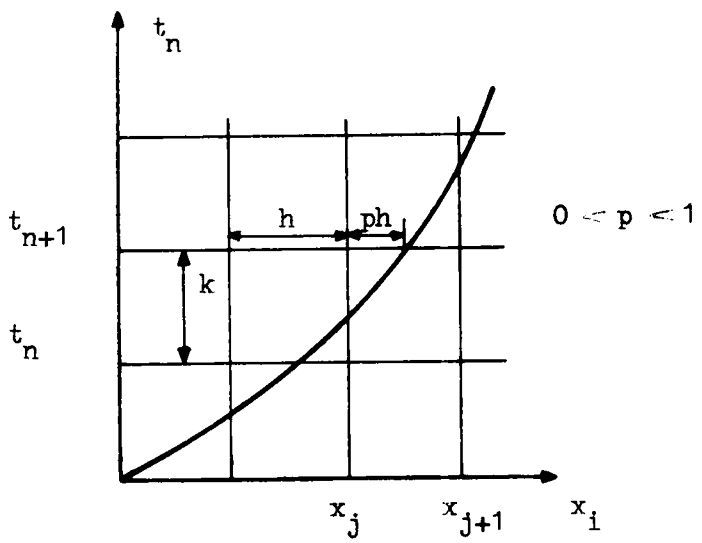

Fig. 3.2 Fixed time step and uniform mesh in space except for the last cell which is variable.

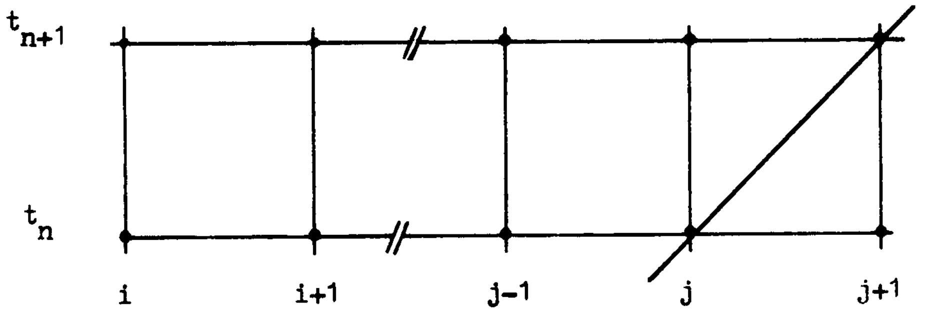

Fig. 3.3 Front position at two consecutive time levels.

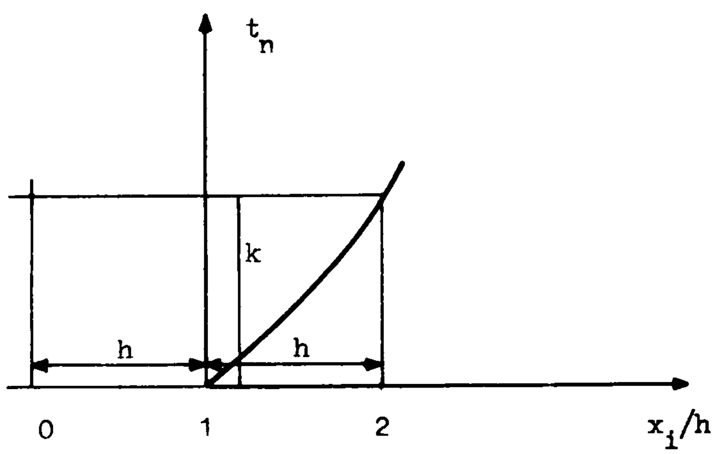

Fig. A1 Scheme for the starting procedure.

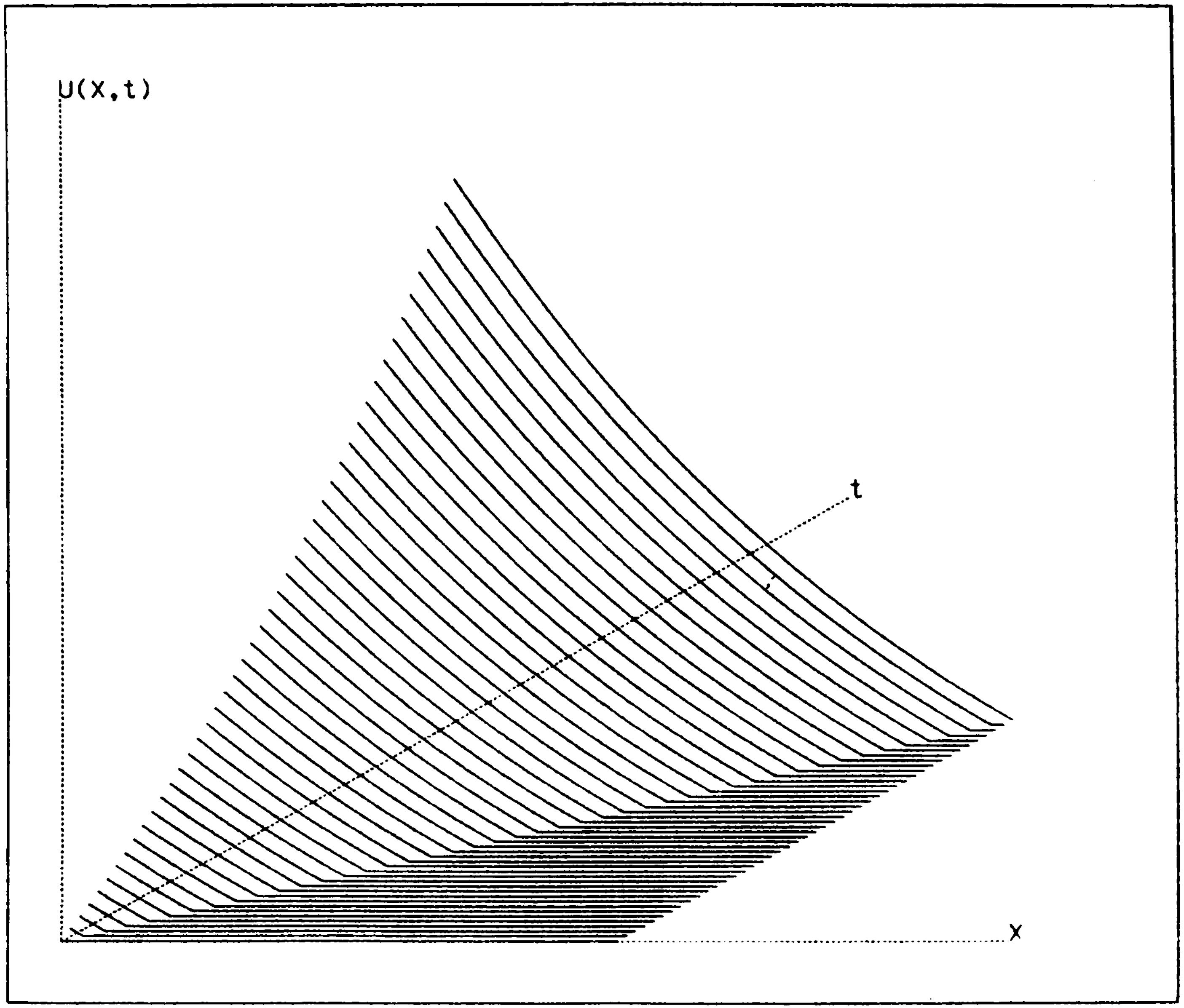

Fig. 4.1 Exact solution in the x-t plane of the straight line front trajectory problem.

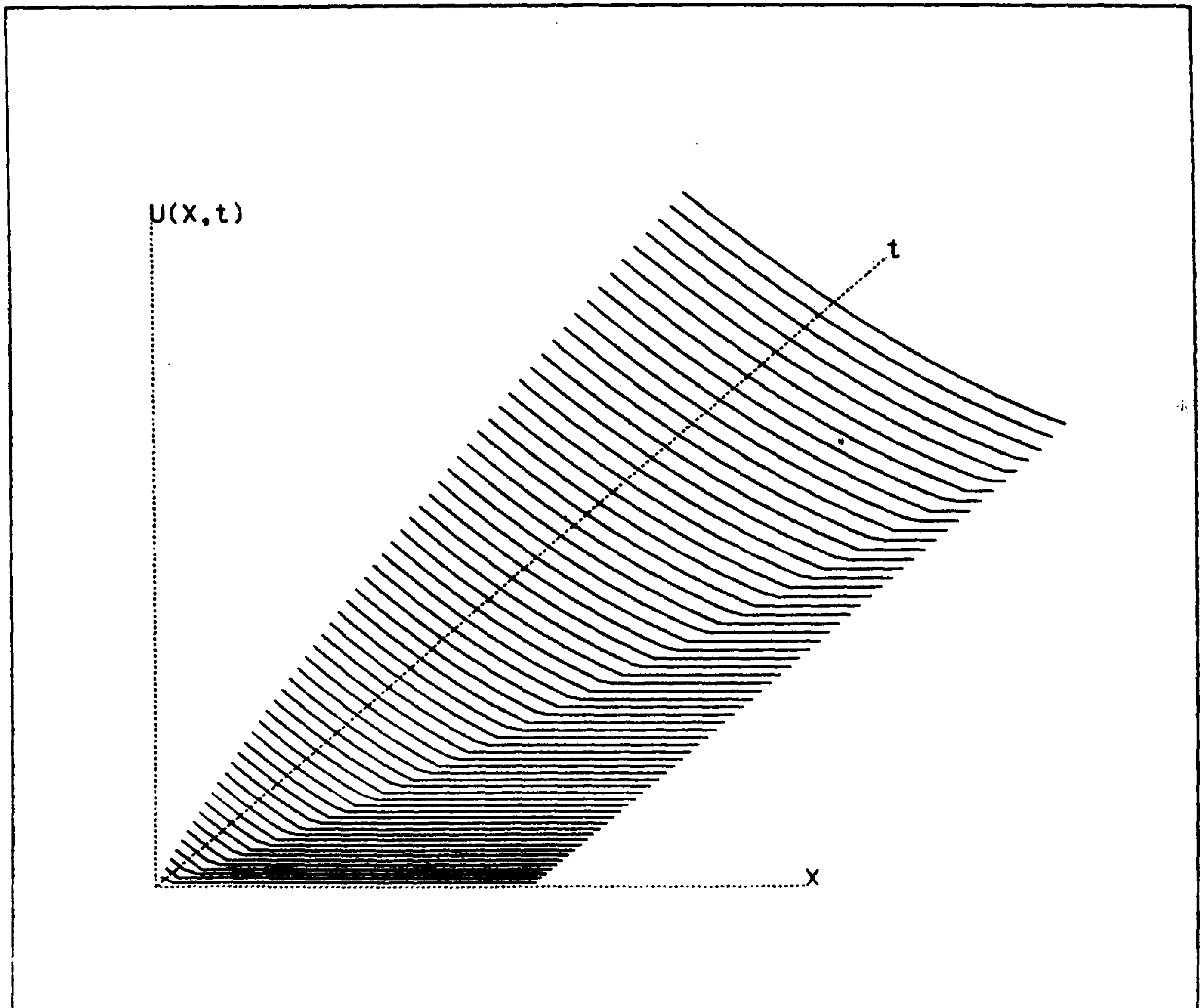

Fig. 4.2 Approximate analytio solution in the x-t plane of the curved line front trajectory problem.