Volume 76B, number 2 PHYSICS LETTERS 22 May 1978

ON THE CONVERGENCE OF THE NUCLEAR FIELD THEORY PERTURBATION EXPANSION FOR STRONGLY ANHARMONIC SYSTEMS

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Received 29 March 1978

The convergence of the nuclear field theory is studied for situations in which the interaction energy between the phonons is of the same order of magnitude as the correlation energy of the phonons.

The nuclear field theory (NFT) (cf., e.g. ref. [1] and references therein; cf. also ref. [2]) has been shown to provide a systematic method to deal with the interaction between the collective and the single-particle degrees of freedom, which are utilized as building blocks of the nuclear spectrum. In this note we study the convergence properties of the partial summations implied by the NFT by comparing the results of the corresponding perturbative expansion with exact results.

We consider systems with four particles or four holes away from a closed shell moving both in a two-level and in a many-level space ‡1 and interacting via a pairing force of constant matrix elements. In zeroth order, the corresponding states are described in terms of two pairing vibrational bosons [3]. The NFT solutions are worked out in what follows up to third order in the perturbation parameter $1/\Omega$, where Ω is the degeneracy of the shell.

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‡1 The use of the two-level model is justified on the basis of transparency, while the solutions of the many-level model are required to dispel the fear of triviality.

The exact solutions of the two-level model can be expressed as analytic functions of G (pairing coupling constant) and of the level degeneracy $\Omega = j + 1/2$ (cf. ref. [4]). In particular the interaction energy (anharmonicity) ΔW between two-pair addition as well as between two-pair removal phonons is given by

$$\Delta W = 2G \,. \tag{1}$$

Utilizing the quantities

$$\begin{aligned} &\epsilon = \epsilon_0 + G: && \text{Hartree–Fock energy,} \\ &\epsilon_0: && \text{distance between the two levels,} \\ &W_0 = \epsilon - G\Omega: && \text{phonon energy ,} \\ &\Lambda = G\sqrt{\Omega}: && \text{particle-vibration coupling strength,} \end{aligned} \tag{2}$$

which completely define the fermion and boson fields, the NFT graphs displayed in fig. 1 are calculated. In the framework of the Rayleigh–Schrödinger (RS) perturbation theory we obtain

$$\begin{aligned} &\text{graph 1a} && = 2G && 1/\Omega \,, \\ &\text{graphs 1b–1f} && = -10G/\Omega \\ &\text{graph 1g} && = 10G/\Omega \end{aligned} \Big\} \; 1/\Omega^2 \,. \tag{3}$$

Thus, the lowest order contribution $(1/\Omega)$ already gives the exact result. Note the exact cancellation between

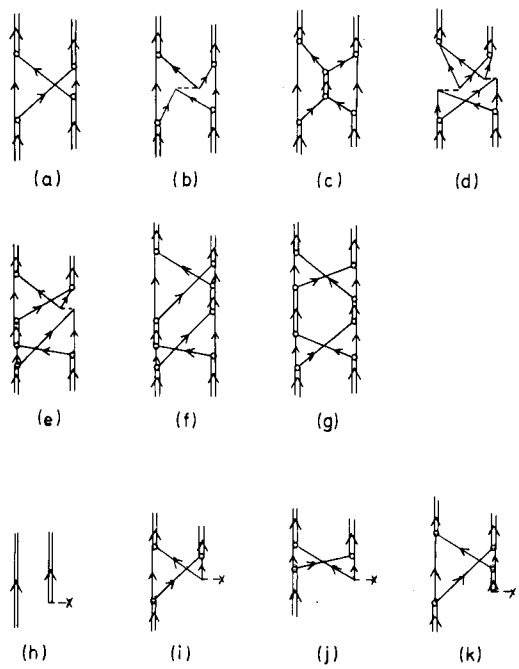

Fig. 1. The graphs 1a–1g are the NFT contributions (in RS perturbation theory) up to order $(1/\Omega)^2$ to the pair addition phonon–phonon interaction. The same graphs are obtained for the case of the pair removal mode by inverting the sense of the single- and double-arrowed lines. The diagrams 1h–1k are the NFT contributions, up to order $(1/\Omega)$, to the matrix element of the two-body transfer operator between one- and two-phonon states. In the many-level case, other time orderings than the ones displayed have to be considered, because of the existence of non-collective phonons. Thus, e.g. graphs with two-phonons in intermediate states are allowed.

the two large contributions of order $(1/\Omega)^2$. A similar cancellation is also found in order $(1/\Omega)^3$, in which case the two contributions of opposite sign have a value equal to $50\,G/\Omega^2$.

The diagrams h–k of fig. 1 display the NFT contributions, up to order $1/\Omega$, to the matrix element of the two-body transfer operator T between one- and two-phonon states. To obtain the corresponding transition amplitudes, this matrix element has to be multiplied by the amplitude

$$|x_0| = \langle \widetilde{\#\#} \,|\, \#\# \rangle^{1/2} = \left(1 + \frac{\mathrm{d}(\Delta W)}{\mathrm{d}(2W_0)}\right)^{1/2} . \qquad (4)$$

Here $|\#\#\rangle$ is the unperturbed two-phonon state while $|\widetilde{\#\#}\rangle$ is the perturbed (normalized) state. For more details cf. e.g. ref. [5].

Table 1
In (a) are reported the results of the diagonalization and of the NFT for the interaction energies associated with the two-pair addition (gs(^{212}Pb)) and two-pair removal (gs(^{204}Pb)) phonons. The NFT contributions of order $1/\Omega$, $1/\Omega^2$ and $1/\Omega^3$ were calculated in the framework of the RS perturbation theory (cf. fig. 1). In (b) we display the ratios

$$R_1 = \frac{\sigma(\mathrm{gs}(^{206}\mathrm{Pb}) \to \mathrm{gs}(^{204}\mathrm{Pb}))}{\sigma(\mathrm{gs}(^{208}\mathrm{Pb}) \to \mathrm{gs}(^{206}\mathrm{Pb}))},$$

and

$$R_2 = \frac{\sigma(\mathrm{gs}(^{210}\mathrm{Pb}) \to \mathrm{gs}(^{212}\mathrm{Pb}))}{\sigma(\mathrm{gs}(^{208}\mathrm{Pb}) \to \mathrm{gs}(^{210}\mathrm{Pb}))}$$

(cf. eq. (5)), calculated utilizing the results of the diagonalization and the $1/\Omega$ NFT diagrams calculated in the RS perturbation theory.

Order	ΔW (keV)	
	I	II
$1/\Omega$	474	262
$1/\Omega^2$	42	4.3
$1/\Omega^3$	14	0.4
	530	266.7
Exact	540	267

(a)

	Exact	NFT
R_1	1.797	1.724
R_2	1.909	1.908

(b)

Keeping contributions up to order $1/\Omega$ we obtain

$$\sigma \equiv |\langle \widetilde{\#\#} | T | \#\# \rangle|^2 = |x_0|^2 \, |\langle \#\# | T | \#\# \rangle|^2$$
$$= 2(1 - 1/\Omega) , \qquad (5)$$

where the usual bosonic factor of two is modified by the Pauli principle correction. Again, already the order $1/\Omega$ coincides with the exact result.

If the partial summations implied by the NFT produce a rapidly convergent expansion in the framework of the RS perturbation theory, this is not so within the framework of the Brillouin–Wigner (BW) perturbation theory. This can be seen starting with the secular equation (cf. ref. [4])

$$E - 2W_0 = W(E) , \qquad (6)$$

which is obtained summing the BW expression of the NFT diagrams [‡2] to infinite order in $1/\Omega$. This sum gives rise to a geometrical series, i.e.

$$W(E) = B(E)/(1 + A(E)) . \tag{7}$$

The functions $B(E)$ and $A(E)$ can be written, for $E = E_{\text{exact}} = 2W_0 + 2G$, as

$$\begin{aligned} B(E_{\text{exact}}) &= 2\Omega^3 G/[(\Omega-2)^2(\Omega-1)] , \\ A(E_{\text{exact}}) &= (5\Omega^2 - 8\Omega + 4)/[(\Omega-2)^2(\Omega-1)] . \end{aligned} \tag{8}$$

Note that the values of $A(E_{\text{exact}})$ are, for $3 < \Omega < 13$, very large (12.5–0.51). Thus, for this range of degeneracies which span the physically interesting region, the function $W(E)$ cannot be calculated by a few-term expansion of the right-hand side of eq. (6).

We discuss now as examples of many-shell systems the ground state of ^{204}Pb and ^{212}Pb. As mentioned, these states can be described in zeroth order as two-phonon pairing vibrational states, i.e.

$$\begin{aligned} |\mathrm{I}\rangle &\equiv |\mathrm{gs}(^{204}\mathrm{Pb})\rangle = |\mathrm{gs}(^{206}\mathrm{Pb}) \otimes \mathrm{gs}(^{206}\mathrm{Pb})\rangle , \\ |\mathrm{II}\rangle &\equiv |\mathrm{gs}(^{212}\mathrm{Pb})\rangle = |\mathrm{gs}(^{210}\mathrm{Pb}) \otimes \mathrm{gs}(^{210}\mathrm{Pb})\rangle . \end{aligned} \tag{9}$$

Note that the "rest mass" (correlation energy) of the pair removal mode (gs(^{206}Pb)) is 640 keV, to be compared with the experimental value of the phonon–phonon interaction energy

$$B(^{204}\mathrm{Pb}) - B(^{208}\mathrm{Pb}) - 2\,[B(^{206}\mathrm{Pb}) - B(^{208}\mathrm{Pb})] = 706 \text{ keV} , \tag{10}$$

where $B(A)$ is the binding energy of the nucleus A. In the case of ^{212}Pb the corresponding numbers are 1237 keV and 169 keV, respectively. We can thus expect convergence problems in the case of ^{204}Pb.

The exact results for the interaction energy of the two systems and for the matrix element of the two-particle transfer operator were obtained diagonalizing the pairing interaction in a basis of four holes (particles), moving around the $N = 126$ closed shell. The levels included in the calculation were taken from experiment and are $p_{1/2}$(0.0 MeV), $f_{5/2}$(−0.57 MeV), $p_{3/2}$(−0.9 MeV), $i_{13/2}$(−1.64 MeV), $f_{7/2}$(−2.35 MeV), $h_{9/2}$(−3.47 MeV), $h_{11/2}$(−9.07 MeV) for ^{204}Pb and $g_{9/2}$(3.432 MeV), $i_{11/2}$(4.208 MeV), $j_{15/2}$(4.853 MeV), $d_{5/2}$(4.998 MeV), $s_{1/2}$(5.467 MeV), $g_{7/2}$(5.925 MeV), $d_{3/2}$(5.973 MeV) for the case of ^{212}Pb. The pairing strengths needed to reproduce the correlation energy of gs(^{206}Pb) (−640 keV) and of gs(^{210}Pb) (−1237 keV) are $G = 0.13$ MeV and $G = 0.10$ MeV, respectively.

The resulting values of the interaction energy for the states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ are

$$\Delta W_{\mathrm{I}} = 540 \text{ keV} , \quad \Delta W_{\mathrm{II}} = 267 \text{ keV} . \tag{11}$$

The values (11) thus play the role played by eq. (2) in the two-level model. The NFT results, calculated using RS perturbation theory are displayed in table 1. In the case of ^{204}Pb, already the $1/\Omega$ contribution accounts for 90% of the exact value. The $1/\Omega^2$ and $1/\Omega^3$ contributions amount to 8% and 3% of the $1/\Omega$ contribution. Although these values indicate a somewhat slow convergence, they arise from cancellations between large direct contributions and wave function renormalization type diagrams, as already discussed for the two-level model. In fact

$$\begin{aligned} \Delta W_{\mathrm{I}}(1/\Omega^2) &= (-0.727 + 0.769) \text{ MeV} = 42 \text{ keV} , \\ \Delta W_{\mathrm{I}}(1/\Omega^3) &= (1.155 - 1.141) \text{ MeV} = 14 \text{ keV} . \end{aligned} \tag{12}$$

For the case of ^{212}Pb the convergence is, as expected, extremely fast.

In table 1b we collect the exact and the NFT (up to order $(1/\Omega)$) results for the two-particle transfer operator. They display the same-kind of agreement as the interaction energies shown in table 1a.

We conclude that the NFT prescription for summing up the contributions to the different physical quantities, within the framework of the RS perturbation theory, is strongly convergent. Thus, the lowest-order contributions in the parameter $1/\Omega$ give a rather accurate approximation to the exact solution.

Discussions with A. Bohr, C. Mahaux, M. Vincent, K. Matsuyanagi and B.R. Mottelson are gratefully acknowledged. One of us (PFB) would like to acknowledge the financial support from the Danish Research Council.

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