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N-N INTERACTION MATRIX ELEMENTS FROM PHASE SHIFTS AND THE JOST FUNCTION

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Abstract: An improvement of the Born approximation based on properties of the Jost function is tested by calculating N-N potential matrix elements from experimental phase shifts. Satisfactory results are obtained with a simple plane-wave method even for the S-channel.

1. Introduction

In recent years, a number of methods [1-6]) have been developed to evaluate the matrix elements of the nucleon-nucleon interaction directly from the experimental elastic data, avoiding the introduction of "realistic" potentials with parameters to determine. In all cases the matrix elements have been calculated in a basis of harmonic oscillator wave functions (HOF). If matrix elements in any other basis are needed they can be written in terms of the HOF.

With the exception of the Sussex group calculation [5]) the matrix elements have been estimated using the plane-wave Born approximation (BA) and therefore they are seriously restricted in the S-wave case. In order to be able to perform the calculation for any l, Elliott *et al.* [5]) made use of the auxiliary-potential method in the framework of the distorted-wave Born approximation (DWBA). In ref. [7]) a correction to the BA and DWBA was proposed, based in the Jost function theory. The aim of this paper is to apply this correction to the calculation of nucleon-nucleon interaction matrix elements. In the following, BA* and DWBA* will denote the "corrected" first Born approximations.

Following closely the scheme developed in ref. [4]) we evaluate in sect. 2 the matrix elements for uncoupled channels in both BA and BA*. The method is generalized to coupled channels in sect. 3 and in sect. 4 the correction is applied to the auxiliary-potential method. The results are finally discussed in sect. 5 while in the appendix the formulae for BA and BA* are deduced.

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2. Uncoupled channels

For each channel $\alpha = {}^{2s+1}l_j$ the matrix elements can be written as a function of the Talmi integrals

$$\langle n, slj|V_\alpha|n', slj\rangle = \sum_{\mu=0}^{n+n'} B(nl, n'l, l+\mu)I(l, \mu), \tag{1}$$

where the coefficients B are given in ref. [8]) and

$$I(l, \mu) = \frac{2}{\Gamma(l+\mu+\frac{3}{2})}\int_0^\infty V_\alpha(r)\mathrm{e}^{-(r/q)^2}\left(\frac{r}{q}\right)^{2l+2\mu+2}\frac{\mathrm{d}r}{q}, \qquad q = \sqrt{2}b, \tag{2}$$

where $b = (\hbar/M\omega)^{\frac{1}{2}}$ is the harmonic oscillator size parameter. As in ref. [4]) we can calculate the Talmi integrals by an iterative procedure

$$\sum_{\nu=0}^{\beta}(-)^\nu \frac{\mu!l!2^\nu(2l+2\mu+1-2\nu)!!}{(\mu-\nu)!(l-\nu)!\nu!(2l+2\mu+1)!!} I(l, \mu-\nu) = A(l, \mu), \qquad \beta = \min\{l, \mu\}, \tag{3}$$

where the numbers A are obtained as integrals over the free-wave matrix elements

$$A(l, \mu) = \frac{2\hbar\omega}{\pi}\frac{(2\mu+1)!!}{2l+1}\frac{(2l+1)!!}{(2l+2\mu+1)!!}\int_0^\infty \mathrm{e}^{-E}\Delta_\alpha(E)\,{}_1F_1(-l-1-\mu, \tfrac{1}{2}-l; E)\mathrm{d}E, \tag{4}$$

$$0 \leqq \mu \leqq (n+n'),$$

$$E = \frac{E_{\mathrm{lab}}}{\hbar\omega},$$

where

$$\Delta_\alpha(E) = -\frac{\pi}{2k}\langle\alpha, k|V|\alpha, k\rangle = -\frac{Mk}{\hbar^2}\int_0^\infty j_l(kr)V_\alpha(r)j_l(kr)r^2\mathrm{d}r. \tag{5}$$

For the calculation of these free-wave matrix elements Srivastava *et al.* [4]) used the BA (A.12)

$$\Delta_\alpha = \mathrm{tg}\,\delta_\alpha. \tag{6}$$

We also use the BA* (A.14) suggested in ref. [7])

$$\Delta_\alpha(E) = \prod_n\left(1-\frac{E_n}{E}\right)\sin[\delta_\alpha(E)]\exp\mathscr{H}[\delta_\alpha(E)], \tag{7}$$

with

$$\mathscr{H}[f(x)] = \frac{1}{\pi}\int_0^\infty \frac{f(x')}{x-x'}\mathrm{d}x', \tag{8}$$

and $E_n^\alpha < 0$ refers to the energies of the bound states, if any. Approximations (6) and (7) are equivalent for $\delta_\alpha \approx 0$ but the BA tends to differ from the better BA* as δ_α becomes larger.

TABLE I

The matrix elements $\langle n, {}^1S_0|V|n', {}^1S_0\rangle$ in MeV

b =		1.4 fm		1.6 fm		1.8 fm		2.0 fm		2.2 fm		2.4 fm		2.6 fm	
n	n'	BA*	BA	BA*	BA	BA*	BA	BA*	BA	BA*	BA	BA*	BA	BA*	BA
0	0	−10.09	−11.27	−7.80	−10.49	−6.06	−9.56	−4.75	−8.64	−3.78	−7.79	−3.04	−7.02	−2.49	−6.33
0	1	−7.35	−1.79	−6.55	−2.92	−5.53	−3.44	−4.58	−3.64	−3.77	−3.66	−3.11	−3.57	−2.58	−3.44
0	2	−4.06	1.68	−4.69	0.07	−4.50	−0.87	−4.02	−1.43	−3.49	−1.74	−2.99	−1.90	−2.54	−1.96
0	3	−1.01	3.34	−2.79	1.54	−3.31	0.45	−3.29	−0.24	−3.05	−0.69	−2.72	−0.99	−2.40	−1.18
1	1	−5.50	−5.28	−5.67	−5.61	−5.23	−5.54	−4.61	−5.30	−3.99	−5.00	−3.42	−4.65	−2.92	−4.28
1	2	−2.85	0.07	−4.01	−1.48	−4.18	−2.23	−3.94	−2.61	−3.55	−2.81	−3.13	−2.93	−2.75	−3.03
1	3	−0.25	3.01	−2.35	0.87	−3.13	−0.29	−3.27	−0.94	−3.12	−1.28	−2.85	−1.43	−2.52	−1.47
2	2	−1.09	−1.90	−2.93	−3.06	−3.56	−3.48	−3.62	−3.54	−3.42	−3.39	−3.10	−3.07	−2.72	−2.63
2	3	1.03	1.97	−1.43	−0.16	−2.50	−1.26	−2.87	−1.90	−2.92	−2.35	−2.83	−2.73	−2.68	−3.06
3	3	2.32	0.73	−0.50	−1.19	−1.91	−1.98	−2.47	−2.17	−2.59	−1.97	−2.47	−1.58	−2.22	−1.09
		The matrix elements $\langle n, {}^3P_0|V|n', {}^3P_0\rangle$ in MeV													
0	0	−3.12	−2.99	−2.43	−2.43	−1.82	−1.89	−1.36	−1.45	−1.02	−1.11	−0.77	−0.85	−0.59	−0.66
0	1	−1.49	−1.10	−1.70	−1.46	−1.52	−1.41	−1.25	−1.23	−1.01	−1.03	−0.80	−0.84	−0.64	−0.69
0	2	0.26	0.57	−0.74	−0.42	−1.00	−0.77	−0.97	−0.84	−0.85	−0.79	−0.72	−0.71	−0.60	−0.61
0	3	1.72	1.75	0.16	0.41	−0.46	−0.21	−0.65	−0.47	−0.66	−0.55	−0.60	−0.54	−0.53	−0.50
1	1	−0.81	−0.66	−1.52	−1.29	−1.57	−1.40	−1.40	−1.31	−1.17	−1.14	−0.96	−0.97	−0.78	−0.81
1	2	0.85	0.80	−0.60	−0.42	−1.09	−0.89	−1.16	−1.01	−1.07	−0.98	−0.93	−0.89	−0.78	−0.77
1	3	2.43	2.21	0.44	0.51	−0.46	−0.27	−0.78	−0.60	−0.85	−0.72	−0.80	−0.72	−0.72	−0.68
2	2	1.78	1.47	−0.01	−0.01	−0.79	−0.64	−1.03	−0.88	−1.03	−0.92	−0.95	−0.87	−0.83	−0.80
2	3	2.96	2.59	0.86	0.74	−0.24	−0.17	−0.71	−0.57	−0.86	−0.72	−0.85	−0.75	−0.77	−0.71
3	3			1.44	1.20	0.16	0.14	−0.47	−0.38	−0.74	−0.62	−0.81	−0.70	−0.79	−0.71
		The matrix elements $\langle n, {}^1P_1|V|n', {}^1P_1\rangle$ in MeV													
0	0	3.57	2.57	1.87	1.33	1.05	0.75	0.64	0.46	0.42	0.31	0.30	0.22	0.22	0.16
0	1	5.31	3.96	2.92	2.09	1.62	1.14	0.92	0.65	0.55	0.39	0.35	0.25	0.24	0.18
0	2	6.16	4.93	3.75	2.76	2.20	1.56	1.29	0.90	0.76	0.53	0.47	0.33	0.30	0.21
0	3	6.07	5.41	4.17	3.24	2.65	1.92	1.63	1.15	1.00	0.69	0.61	0.43	0.38	0.27
1	1	7.17	5.82	4.45	3.32	2.69	1.92	1.62	1.14	0.99	0.70	0.63	0.44	0.41	0.29
1	2	7.41	6.72	5.08	4.03	3.29	2.43	2.08	1.48	1.32	0.92	0.84	0.59	0.55	0.39
1	3	7.00	7.19	5.28	4.53	3.66	2.82	2.43	1.77	1.58	1.12	1.03	0.72	0.68	0.48
2	2	7.60	7.78	5.71	4.91	3.96	3.07	2.65	1.94	1.74	1.24	1.15	0.81	0.76	0.54

The matrix elements $\langle n, {}^3P_1|V|n', {}^3P_1\rangle$ in MeV

0	0	4.59	3.46	2.98	2.21	2.03	1.48	1.43	1.04	1.04	0.75	0.77	0.55	0.58	0.42
0	1	4.64	3.70	3.06	2.35	2.12	1.58	1.52	1.12	1.12	0.82	0.85	0.61	0.65	0.47
0	2	4.44	3.81	2.97	2.38	2.06	1.59	1.48	1.12	1.10	0.82	0.84	0.62	0.65	0.47
0	3	4.18	3.93	2.86	2.42	2.00	1.60	1.44	1.11	1.07	0.81	0.81	0.61	0.63	0.47
1	1	6.15	5.16	4.08	3.22	2.80	2.14	1.99	1.49	1.46	1.08	1.10	0.80	0.84	0.61
1	2	6.25	5.68	4.26	3.53	2.97	2.34	2.13	1.63	1.57	1.18	1.19	0.88	0.92	0.67
1	3	5.90	5.95	4.18	3.68	2.95	2.42	2.14	1.68	1.59	1.21	1.21	0.91	0.94	0.70
2	2	6.97	6.85	4.88	4.24	3.43	2.79	2.46	1.93	1.82	1.39	1.37	1.03	1.06	0.79
2	3	6.75	7.41	4.99	4.62	3.58	3.03	2.60	2.09	1.93	1.50	1.45	1.11	1.12	0.84
3	3			5.33	5.25	3.90	3.43	2.85	2.35	2.13	1.69	1.63	1.26	1.27	0.96

The matrix elements $\langle n, {}^1D_2|V|n', {}^1D_2\rangle$ in MeV

0	0	−1.17	−1.30	−0.68	−0.76	−0.42	−0.46	−0.27	−0.30	−0.18	−0.20	−0.12	−0.13	−0.08	−0.09
0	1	−1.25	−1.37	−0.75	−0.83	−0.47	−0.52	−0.30	−0.34	−0.21	−0.23	−0.14	−0.16	−0.10	−0.11
0	2	−1.21	−1.29	−0.74	−0.81	−0.47	−0.52	−0.31	−0.35	−0.21	−0.24	−0.15	−0.17	−0.11	−0.12
0	3	−1.12	−1.16	−0.71	−0.76	−0.46	−0.51	−0.31	−0.34	−0.21	−0.23	−0.15	−0.17	−0.11	−0.12
1	1	−1.77	−1.90	−1.07	−1.17	−0.67	−0.75	−0.44	−0.49	−0.30	−0.33	−0.21	−0.23	−0.15	−0.16
1	2	−1.86	−1.96	−1.15	−1.25	−0.74	−0.82	−0.49	−0.54	−0.33	−0.37	−0.23	−0.26	−0.17	−0.19
1	3	−1.80	−1.84	−1.15	−1.23	−0.75	−0.82	−0.50	−0.56	−0.35	−0.39	−0.24	−0.27	−0.18	−0.20
2	2	−2.22	−2.28	−1.39	−1.49	−0.90	−0.98	−0.60	−0.66	−0.41	−0.46	−0.29	−0.32	−0.21	−0.23
2	3			−1.47	−1.55	−0.97	−1.04	−0.65	−0.72	−0.45	−0.50	−0.32	−0.35	−0.23	−0.26
3	3			−1.66	−1.72	−1.10	−1.18	−0.75	−0.81	−0.52	−0.57	−0.37	−0.41	−0.27	−0.30

The matrix elements $\langle n, {}^3D_2|V|n', {}^3D_2\rangle$ in MeV

0	0	−4.38	−6.11	−2.68	−3.86	−1.71	−2.50	−1.12	−1.65	−0.76	−1.12	−0.52	−0.77	−0.37	−0.54
0	1	−4.19	−5.34	−2.69	−3.70	−1.78	−2.55	−1.21	−1.78	−0.85	−1.25	−0.60	−0.89	−0.44	−0.65
0	2	−3.61	−4.00	−2.43	−3.07	−1.66	−2.26	−1.16	−1.65	−0.83	−1.20	−0.60	−0.89	−0.44	−0.66
0	3	−3.02	−2.74	−2.13	−2.40	−1.50	−1.90	−1.07	−1.45	−0.77	−1.10	−0.57	−0.83	−0.43	−0.63
1	1	−5.69	−6.81	−3.70	−4.83	−2.46	−3.39	−1.68	−2.40	−1.18	−1.71	−0.84	−1.24	−0.61	−0.90
1	2	−5.54	−5.96	−3.75	−4.57	−2.57	−3.38	−1.79	−2.48	−1.27	−1.82	−0.92	−1.34	−0.68	−1.00
1	3	−4.86	−4.47	−3.48	−3.85	−2.45	−3.03	−1.74	−2.32	−1.26	−1.76	−0.93	−1.33	−0.69	−1.01
2	2	−6.32	−6.40	−4.39	−5.07	−3.03	−3.84	−2.13	−2.86	−1.52	−2.12	−1.11	−1.59	−0.82	−1.19
2	3			−4.39	−4.70	−3.11	−3.73	−2.22	−2.87	−1.61	−2.19	−1.19	−1.67	−0.89	−1.28
3	3			−4.77	−4.84	−3.45	−3.93	−2.48	−3.10	−1.81	−2.40	−1.34	−1.84	−1.00	−1.43

Our lack of knowledge of the phase shifts for energies larger than 460 MeV does not allow the exact evaluation of the integrals (4). However one can confidently carry out the integration up to an energy E_f due to the presence of the exponential function, provided that at the same time one imposes an upper limit to μ for each channel so as to avoid the effect of the hypergeometric function ${}_1F_1$ at high energies.

In table 1 we present the results for the matrix elements in both BA and BA*. In all the cases considered we found good agreement between the results with the

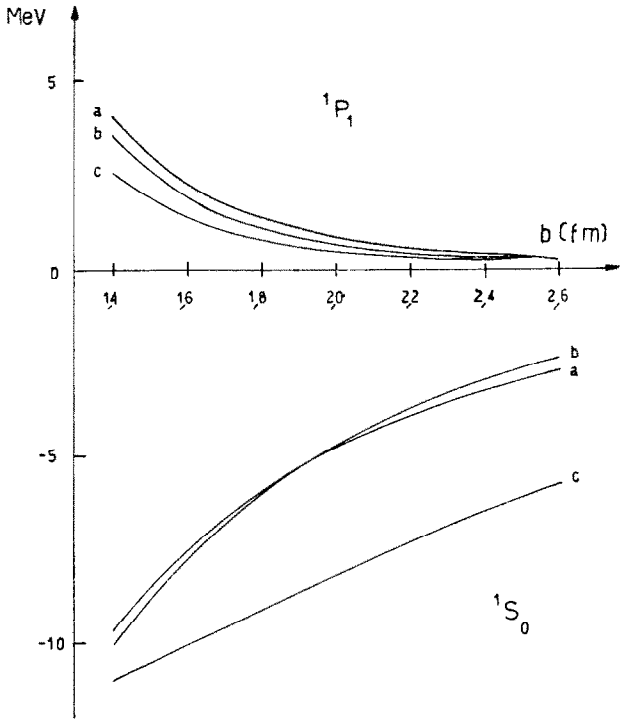

Fig. 1. The matrix elements $\langle n=0, {}^1P_1|V|n'=0, {}^1P_1\rangle$ and $\langle n=0, {}^1S_0|V|n'=0, {}^1S_0\rangle$: a) the results of Elliott *et al.* [5]), b) present work with the "corrected" BA* and c) with the usual BA.

BA* and those obtained with the DWBA quoted in ref. [5]), even for the ${}^1S_0$ channel where the usual BA fails. The experimental phase shifts used are those of ref. [9]). In fig. 1 we show our results for BA* and BA and compare with those of the DWBA. There is no point in considering small differences between our results and those of ref. [5]) since they have been obtained from different sets of phase shifts.

3. Coupled channels

The channels ${}^3(j-1)_j$ and ${}^3(j+1)_j$ are coupled by the N-N interation. For the channel diagonal matrix elements the method of sect. 2 can be used, with the free

wave matrix elements given by

$$\Delta^{\text{on}}_{j\pm1} = -\frac{\pi}{2k}\langle s, j\pm1, j|V|s, j\pm1, j\rangle$$
$$= -\frac{Mk}{\hbar^2}\int_0^\infty r^2\mathrm{d}r\int \mathrm{d}\Omega j_{j\pm1}(kr)\mathscr{Y}^{*m}_{sj\pm1j}V\mathscr{Y}^{m}_{sj\pm1j}j_{j\pm1}(kr). \quad (9)$$

The group of the University of McMaster [4]) has also developed a method to evaluate from the channel off-diagonal matrix elements in the free-wave basis

$$\Delta^{\text{off}}_j = -\frac{\pi}{2k}\langle s, j-1, j|V|s, j+1, j\rangle = -\frac{\pi}{2k}\langle s, j+1, j|V|s, j-1, j\rangle, \quad (10)$$

those corresponding to the HOF basis.

We begin by evaluating the first Talmi integral

$$I(j,0) = \frac{2\hbar\omega}{\pi}\int_0^\infty \mathrm{e}^{-E}\Delta^{\text{off}}_j(E)$$
$$\times\left[\frac{1}{2j+1}\,{}_1F_1(-j-1,\,-j+\tfrac{1}{2},\,E) - \frac{1}{E}\,{}_1F_1(-j-1,\,-j-\tfrac{1}{2},\,E)\right]\mathrm{d}E, \quad (11)$$

$$A(j,\mu) = \frac{2\hbar\omega}{\pi}\,\frac{(2\mu+3)!!(2j+1)!!}{(2j-1)(2j+1)(2j+2\mu+1)!!}$$
$$\times\int_0^\infty \mathrm{e}^{-E}\Delta^{\text{off}}_j(E)\,{}_1F_1(-j-1-\mu,\,\tfrac{3}{2}-j,\,E)E\,\mathrm{d}E. \quad (12)$$

We then calculate

$$H_{\text{T}}(j,0) = A(j,0),$$
$$H_{\text{T}}(j,\mu) = A(j,\mu) + \sum_{\nu=1}^{\beta}\frac{(-)^{\nu+1}2^\nu\mu!\,j!(2j+2\mu+1-2\nu)!!}{(\mu-\nu)!(j-\nu)!\nu!(2j+2\mu+1)!!}H_{\text{T}}(j,\mu-\nu) \quad (13)$$

from which the other Talmi integrals can be obtained by means of

$$I(j,\mu+1) = \frac{1}{j+\mu+\frac{3}{2}}\left[(j+\mu+\tfrac{1}{2})I(j,\mu) - H_{\text{T}}(j,\mu)\right] \quad (14)$$

and finally the matrix elements in the HOF basis by

$$\langle n, j\pm1, j|V|n', j\mp1, j\rangle = \sum_{\mu=0}^{n+n'}B(nj\pm1, n'j\mp1, j+\mu)I(j,\mu). \quad (15)$$

Again, in order to apply this formalism one has to obtain the free-wave matrix elements (9) and (10) from elastic data, i.e., the eigenphase shifts $\delta_{j\pm1}$ and the coupling parameter ε_j.

TABLE 2

The matrix elements $\langle n, {}^3S_1|V|n', {}^3S_1\rangle$ in MeV

b =		1.4 fm		1.6 fm		1.8 fm		2.0 fm		2.2 fm		2.4 fm		2.6 fm	
n	n'	BA*	BA	BA*	BA	BA*	BA	BA*	BA	BA*	BA	BA*	BA	BA*	BA
0	0	−6.09	−11.79	−4.66	−10.65	−3.61	−9.59	−2.83	−8.68	−2.26	−7.91	−1.83	−7.25	−1.50	−6.70
0	1	−4.72	−3.23	−4.00	−3.64	−3.30	−3.62	−2.70	−3.43	−2.22	−3.19	−1.83	−2.95	−1.52	−2.74
0	2	3.12	−0.22	−3.11	−1.40	−2.79	−1.88	−2.42	−2.04	−2.06	−2.03	−1.75	−1.94	−1.48	−1.79
0	3	−1.60	1.76	−2.20	0.04	−2.23	−0.77	−2.06	−1.17	−1.84	−1.35	−1.61	−1.42	−1.41	−1.43
1	1	−4.08	−6.91	−3.80	−7.26	−3.31	−7.12	−2.83	−6.80	−2.40	−6.40	−2.04	−5.96	−1.74	−5.49
1	2	−2.65	−0.52	−2.98	−1.96	−2.81	−2.50	−2.51	−2.70	−2.18	−2.78	−1.89	−2.84	−1.64	−2.94
1	3	−1.14	2.15	−2.10	−0.07	−2.29	−1.05	−2.19	−1.43	−1.98	−1.52	−1.75	−1.47	−1.52	−1.35
2	2	−1.65	−3.22	−2.48	−4.84	−2.59	−5.36	−2.44	−5.34	−2.19	−5.01	−1.92	−4.49	−1.65	−3.84
2	3	−0.30	2.36	−1.60	−0.20	−2.03	−1.47	−2.06	−2.17	−1.96	−2.66	−1.81	−3.07	−1.67	−3.42
3	3			−1.03	−2.63	−1.72	−3.67	−1.89	−3.81	−1.83	−3.44	−1.64	−2.83	−1.41	−2.19
		The matrix elements $\langle n, {}^3D_1|V|n', {}^3D_1\rangle$ in MeV													
0	0	4.75	3.80	3.02	2.35	1.96	1.50	1.29	0.98	0.87	0.66	0.59	0.45	0.41	0.31
0	1	4.07	3.51	2.88	2.34	2.01	1.58	1.40	1.08	0.99	0.75	0.70	0.53	0.51	0.38
0	2	2.92	2.84	2.33	2.03	1.76	1.44	1.30	1.03	0.96	0.74	0.70	0.54	0.52	0.39
0	3	1.89	2.19	1.75	1.68	1.44	1.25	1.13	0.93	0.87	0.69	0.66	0.51	0.50	0.38
1	1	5.09	4.60	3.68	3.10	2.62	2.12	1.87	1.47	1.34	1.04	0.97	0.74	0.71	0.54
1	2	4.38	4.32	3.43	3.06	2.59	2.17	1.92	1.55	1.42	1.12	1.06	0.82	0.79	0.60
1	3	3.21	3.62	2.83	2.73	2.28	2.02	1.78	1.49	1.37	1.10	1.05	0.82	0.80	0.62
2	2	4.69	4.85	3.77	3.50	2.90	2.51	2.19	1.81	1.65	1.32	1.24	0.97	0.94	0.72
2	3			3.46	3.41	2.79	2.52	2.18	1.86	1.69	1.38	1.30	1.03	1.00	0.78
3	3			3.57	3.68	2.94	2.74	2.34	2.04	1.83	1.53	1.43	1.15	1.11	0.87
		The matrix elements $\langle n, {}^3S_1|V|n', {}^3D_1\rangle$ in MeV													
0	0	−0.78	−1.68	−0.47	−1.13	−0.29	−0.80	−0.19	−0.58	−0.13	−0.43	−0.09	−0.32	−0.06	−0.24
0	1	−1.21	−2.20	−0.74	−1.50	−0.47	−1.07	−0.31	−0.79	−0.22	−0.60	−0.15	−0.46	−0.11	−0.35
0	2	−1.55	−2.51	−0.95	−1.71	−0.62	−1.23	−0.41	−0.91	−0.29	−0.70	−0.21	−0.54	−0.15	−0.43
0	3	−1.84	−2.75	−1.14	−1.87	−0.74	−1.33	−0.50	−0.99	−0.35	−0.76	−0.25	−0.60	−0.18	−0.47
1	0	−0.57	−0.62	−0.36	−0.44	−0.24	−0.34	−0.17	−0.27	−0.12	−0.22	−0.09	−0.18	−0.06	−0.15
1	1	−1.04	−1.54	−0.64	−1.03	−0.41	−0.72	−0.28	−0.53	−0.20	−0.41	−0.14	−0.32	−0.11	−0.26
1	2	−1.46	−2.19	−0.91	−1.48	−0.58	−1.03	−0.39	−0.75	−0.28	−0.57	−0.20	−0.44	−0.15	−0.35
1	3	−1.79	−2.59	−1.15	−1.80	−0.74	−1.26	−0.50	−0.92	−0.35	−0.69	−0.25	−0.54	−0.19	−0.43
2	0	−0.44	−0.37	−0.28	−0.24	−0.19	−0.17	−0.13	−0.13	−0.10	−0.11	−0.07	−0.09	−0.06	−0.08
2	1	−0.84	−0.87	−0.55	−0.61	−0.36	−0.43	−0.24	−0.32	−0.17	−0.25	−0.13	−0.20	−0.10	−0.17
2	2	−1.18	−1.48	−0.80	−1.06	−0.53	−0.75	−0.36	−0.54	−0.25	−0.41	−0.18	−0.32	−0.14	−0.26
2	3	−1.45	−1.93	−1.03	−1.45	−0.69	−1.03	−0.47	−0.75	−0.33	−0.56	−0.24	−0.43	−0.17	−0.34
3	0	−0.37	−0.31	−0.24	−0.19	−0.15	−0.12	−0.11	−0.08	−0.08	−0.06	−0.06	−0.05	−0.05	−0.05
3	1	−0.63	−0.52	−0.46	−0.41	−0.31	−0.29	−0.21	−0.21	−0.15	−0.16	−0.11	−0.13	−0.08	−0.11
3	2	−0.86	−0.80	−0.67	−0.70	−0.47	−0.52	−0.32	−0.38	−0.23	−0.29	−0.16	−0.22	−0.12	−0.18
3	3	−1.01	2.89	−0.86	−1.03	−0.62	−0.78	−0.43	−0.57	−0.30	−0.43	−0.22	−0.33	−0.16	−0.27

The usual BA is given by [12]) (A.21)

$$\begin{aligned} \Delta_{j\pm1}^{\text{on}} &= \cos^2 \varepsilon_j \operatorname{tg} \delta_{j\pm1} + \sin^2 \varepsilon_j \operatorname{tg} \delta_{j\mp1}, \\ \Delta_j^{\text{off}} &= \tfrac{1}{2} \sin 2\varepsilon_j (\operatorname{tg} \delta_{j-1} - \operatorname{tg} \delta_{j+1}), \end{aligned} \tag{16}$$

and the corresponding BA* (A.22) by

$$\begin{aligned} \Delta_{j\pm1}^{\text{on}} &= \cos^2 \varepsilon_j \prod_n \left(1 + \frac{k_n^{(j\pm1)^2}}{k^2}\right) \sin \delta_{j\pm1} \exp \mathscr{H}[\delta_{j\pm1}] \\ &\quad + \sin^2 \varepsilon_j \prod_n \left(1 + \frac{k_n^{(j\mp1)^2}}{k^2}\right) \sin \delta_{j\mp1} \exp \mathscr{H}[\delta_{j\mp1}], \\ \Delta_j^{\text{off}} &= \tfrac{1}{2} \sin 2\varepsilon_j \left[\prod_n \left(1 + \frac{k_n^{(j-1)^2}}{k^2}\right) \sin \delta_{j-1} \exp \mathscr{H}[\delta_{j+1}] \right. \\ &\quad \left. - \prod_n \left(1 + \frac{k_n^{(j+1)^2}}{k^2}\right) \sin \delta_{j+1} \exp \mathscr{H}[\delta_{j+1}]\right]. \end{aligned} \tag{17}$$

In table 2 we give the matrix elements for the channels $^3S_1 - {}^3D_1$, using again data by McGregor *et al.* [9]). For the BA in (16) $\operatorname{tg}(\delta_{j\pm1})$ was replaced by $\delta_{j\pm1}$ because $\operatorname{tg} \delta_{^3S_1}$ diverges for $E_{\text{lab}} \approx 17$ MeV.

4. The Jost-function approximation and the DWBA

The correction suggested in ref. [7]) is also applicable to the auxiliary-potential method developed by Elliott *et al.* [5]). In that method the N-N potential is broken into two parts $V = V_0 + V_1$, where V_0 is a cut-oscillator.

Then the matrix element is split into

$$\langle V \rangle = \langle V_0 \rangle + \langle V_1 \rangle. \tag{18}$$

Neglecting a long-range correction, the second term of eq. (18) is given by

$$\langle V_1 \rangle \approx \text{const.} \operatorname{tg}(\delta - \delta_0), \tag{19}$$

where δ is the experimental phase shift and δ_0 is the phase shift produced by V_0. The range and the depth of the auxiliary well are adjusted so that the result in eq. (18) is stable. The correction consists in replacing in expression (19), (A.17),

$$\operatorname{tg}(\delta - \delta_0) \to \sin(\delta - \delta_0) \exp \mathscr{H}(\delta - \delta_0); \tag{20}$$

it, thus, corrects the value of $\langle V_1 \rangle$. Then, if $\langle V_1 \rangle$ is small with respect to $\langle V_0 \rangle$, which occurs for $\delta \approx \delta_0$, it will not modify sensibly the value of $\langle V \rangle$.

The n diagonal matrix elements for the $^1S_0$ channel were evaluated with $\langle V_1 \rangle$ given by eq. (19). Once the DWBA result was stable the correction (20), was applied, but, since the stability was always found for $\delta \approx \delta_0$, the matrix elements were not greatly modified as shown in table 3.

TABLE 3

The matrix elements $\langle n, {}^1S_0|V|n, {}^1S_0\rangle$ in MeV for the auxiliary-potential method

n	A	B	C	V_0	δ	δ_0
0	−4.85	−5.19	−4.89	−3.44	0.9470	0.6051
1	−4.03	−4.23	−4.24	−4.16	0.7927	0.7813
2	−2.88	−3.26	−3.27	−3.19	0.5384	0.5291
3	−1.88	−2.29	−2.18	−2.99	0.4020	0.4921

$b = 2.0$ fm; A ≡ DWBA results of Elliott *et al.*, B ≡ DWBA present work, C ≡ corrected DWBA* present work.

5. Conclusions

The Jost-function theory allows the derivation of a correction to the first Born approximation without going into higher orders.

We have attempted to evaluate the improvement introduced by this correction in the calculation of N-N interaction matrix elements from experimental phase shifts, with respect to the usual plane- and distorted-wave Born approximations. In the case of uncoupled channels it was found that the corrected plane-wave Born approximation gives very satisfactory results, in the sense that they are in good agreement with those obtained with the more laborious auxiliary-potential method. In addition to its simplicity the present method has the advantage that, consisting essentially in the calculation of Talmi integrals, all the dependence in n and n' is contained in the Moshinsky B-coefficients. Therefore, once the Talmi integrals $I(l, \mu)$ up to certain μ have been evaluated, all the matrix elements with $n+n' \leqq \mu$ are already given.

For the ${}^3S_1 - {}^3D_1$ coupled channels the matrix elements have been also evaluated with the BA and BA*. The ${}^3D_1$ diagonal matrix elements for both BA and BA* are similar to the ones quoted in ref. [5]), while for the ${}^3S_1$ partial wave the BA results are larger and the BA* numbers smaller than those of Elliott *et al.*. Our off-diagonal ${}^3S_1 - {}^3D_1$ results are an order of magnitude smaller than those of the Sussex group. As it was pointed out in ref. [4]) this discrepancy may be due to the complications arising in the introduction of a coupling auxiliary potential. As it was noted in sect. 4, when we applied the correction discussed here to the DWBA, in the framework of the auxiliary-potential method given in ref. [5]), we only found small unimportant differences.

Any method for the evaluation of the matrix elements rests on the assumption that the potential is not singular, otherwise they would be infinite. This is not a very stringent restriction however, since realistic non-singular potentials have been proposed [15]) which fit the elastic data as well as the hard-core Hamada-Johnson potential [16]). And even for the case of a hard core, it has been shown in ref. [17]) that the DWBA method, which gives results very similar to the BA*, is equivalent to the Scott-Moszkowski separation procedure [18]).

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Appendix

The N-N transition matrix $T(k)$ is related to the potential V by

$$T(k) = V + V\frac{1}{k^2 - H_0}T(k) = V + V\frac{1}{k^2 - H}V, \tag{A.1}$$

$$\hbar = m = 1,$$

where $H = H_0 + V$ is the total Hamiltonian. For uncoupled channels the on-shell matrix element of T is related to the phase shift by

$$\langle k, slj|T(k^2 + i0)|k, s'l'j'\rangle = -\frac{2k}{\pi}\delta_{ss'}\delta_{ll'}\delta_{jj'}\sin\delta(k)e^{i\delta(k)}, \tag{A.2}$$

where the free-wave function in the Schrödinger representation is

$$\langle r|k, slj\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} kr\, j_l(kr)|slj\rangle, \tag{A.3}$$

with the normalization

$$\langle k, slj|k', s'l'j'\rangle = \delta(k - k')\delta_{ss'}\delta_{ll'}\delta_{jj'}. \tag{A.4}$$

The physical wave function $|\psi_k^+, slj\rangle$ is a solution of the Lippmann-Schwinger equation

$$|\psi_k^+\rangle = |k\rangle + \frac{1}{k^2 - H_0 + i0}V|\psi_k^+\rangle, \tag{A.5}$$

where for simplicity we have omitted the channel labels. Using (A.5) and (A.1) it follows that

$$\langle k|V|\psi_k^+\rangle = \langle k|T(k^2 + i0)|k\rangle = -\frac{2k}{\pi}\sin\delta(k)e^{i\delta(k)}. \tag{A.6}$$

From the last expression we can try to derive a direct approximate relation between the elastic data and the matrix elements of the potential in the free-waves basis, namely if we make in (A.6)

$$|\psi_k^+\rangle \approx N|k\rangle, \tag{A.7}$$

we get

$$\langle k|V|k\rangle \approx -\frac{1}{N}\frac{2k}{\pi}\sin\delta(k)e^{i\delta(k)}. \tag{A.8}$$

In order to choose the constant N let us first consider the asymptotic behaviour of $\langle r|\psi_k^+\rangle$:

$$\langle r|\psi_k^+\rangle \approx \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\delta(k)} \sin\,[kr - \tfrac{1}{2}l\pi + \delta(k)]\,, \tag{A.9}$$

or rewriting it as a linear combination of regular and irregular free waves

$$\langle r|\psi_k^+\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\delta(k)} kr[\cos\,\delta(k) j_l(kr) - \sin\,\delta(k) y_l(kr)], \tag{A.10}$$

then a possible choice for N is

$$N = \cos\,\delta(k) e^{i\delta(k)}, \tag{A.11}$$

which gives in (A.8) the usual BA

$$\langle k|V|k\rangle \approx -\frac{2k}{\pi}\,\mathrm{tg}\,\delta(k). \tag{A.12}$$

However, this is not a good choice for N, since it depends on the behaviour of the physical wave function at large distances from the scattering centre, and for a short-range potential the major contribution to (A.6) will come form the behaviour of the wave functions near the origin. Thus a better choice for N in (A.7) would be

$$N = \lim_{r\to 0} \frac{\langle r|\psi_k^+\rangle}{\langle r|k\rangle} = \frac{g_0(k)}{g(k)}\,, \tag{A.13}$$

where $g_0(k)$ and $g(k)$ are the Jost functions [13]) for the Hamiltonians H_0 and H respectively. In ref. [7]) it is shown that with this value of N the first three terms in the expansion of $\langle r|\psi_k^+\rangle$ and $\langle r|k\rangle N$ around the origin are equal. The Jost function for a potential can be obtained from the corresponding phase shift by means of

$$g(k) = \frac{(2l+1)!!}{k^l} \prod_n \left(1 + \frac{k_n^2}{k^2}\right) \exp\,[\mathscr{H}[\delta(k)] + i(\tfrac{1}{2}l\pi - \delta(k))], \tag{A.14}$$

where $-k_n^2$ are the energies of the bound states in that channel, if any, and $\mathscr{H}[f]$ [13]) is the Hilbert transform of the function f. Formulae (A.13) and (A.14) with (A.8) give

$$\langle k|V|k\rangle \approx -\frac{2k}{\pi} \prod_n \left(1 + \frac{k_n^2}{k^2}\right) \sin\,\delta(k) \exp\,\mathscr{H}[\delta(k)], \tag{A.15}$$

which we have called BA* in this work.

For the above derivation it is not necessary to have started with free waves. We can split the potential as $v = v_0 + v_1$ so that the total Hamiltonian is written as $H = H_1 + v_1$ and all formulae can be modified replacing everywhere $v \to v_1$, $H_0 \to H_1$, $\delta \to \delta - \delta_0$ and the Bessel function for the physical wave function of H_1

to get the ordinary DWBA

$$\langle \psi_k^{(1)^+} | V_1 | \psi_k^{(1)^+} \rangle \approx -\frac{2k}{\pi} \operatorname{tg}(\delta - \delta_0) \tag{A.16}$$

and the corresponding DWBA*

$$\langle \psi_k^{(1)^+} | V_1 | \psi_k^{(1)^+} \rangle \approx -\frac{2k}{\pi} \frac{\prod_n (1 + k_n^{(1)^2}/k^2)}{\prod_m (1 + k_m^{(0)^2}/k^2)} \sin(\delta - \delta_0) \exp \mathscr{H}(\delta - \delta_0). \tag{A.17}$$

In the subspace of the ${}^3(j-1)_j$ and ${}^3(j+1)_j$ coupled channels the transition matrix T is written as a function of the eigenphase shifts $\delta_{j\pm1}$ and the coupling parameter ε_j as

$$\langle k, j | T(k^2 + i0) | k, j \rangle = -\frac{2k}{\pi} U^{-1}(k) \begin{pmatrix} \sin \delta_{j-1} e^{i\delta_{j-1}} & 0 \\ 0 & \sin \delta_{j+1} e^{i\delta_{j+1}} \end{pmatrix} U(k), \tag{A.18}$$

where

$$U(k) = \begin{pmatrix} \cos \varepsilon_j & \sin \varepsilon_j \\ -\sin \varepsilon_j & \cos \varepsilon_j \end{pmatrix} \tag{A.19}$$

diagonalizes T. We also have for the physical wave function

$$\langle k, {}^3L_j | V | \psi_k^+, {}^3L_j' \rangle = \langle k, {}^3L_j | T | k, {}^3L_j' \rangle. \tag{A.20}$$

If we again make in this expression the radial part of $|\psi_n, {}^3L_j\rangle$ proportional to that of $|k, {}^3L_j\rangle$ we get with (A.18) that *in the Born approximation* the potential is also diagonalized by $U(k)$. We have then two uncoupled channels with phase shifts $\delta_{j\pm1}$. Therefore if we choose for each channel the constant N given by (A.11) and (A.13) we get for the BA [ref. [12])]

$$\langle k | V | k \rangle = -\frac{2k}{\pi} U^{-1}(k) \begin{pmatrix} \operatorname{tg} \delta_{j-1} & 0 \\ 0 & \operatorname{tg} \delta_{j+1} \end{pmatrix} U(k) \tag{A.21}$$

and for the BA*

$$\langle k | V | k \rangle = -\frac{2k}{\pi} U^{-1}(k) \times \begin{pmatrix} \prod_n (1 + k_n^2/k^2) \sin \delta_{j-1} \exp \mathscr{H}(\delta_{j-1}) & 0 \\ 0 & \prod_n (1 + k_n^2/k^2) \sin \delta_{j+1} \exp \mathscr{H}(\delta_{j+1}) \end{pmatrix} U(k). \tag{A.22}$$

It should be pointed out that this last derivation for the BA* is not exact as it was for the uncoupled case since though we obtained two uncoupled waves, any of them carry a mixture of angular momenta $l = j \pm 1$. However, it is interesting to note that the expressions we get for BA* (A.15) and DWBA* become exact for the case of a separable potential [15]).

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