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On the Construction of Low-Energy Nuclear Potentials.

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Summary. — It is shown that the potentials that can be defined for the nucleon-nucleon problem from the theory of double dispersion relations, cannot be constructed by a generalization of the method of Charap and Fubini. An explicit expression for the identification of the potentials in the nonrelativistic limit in configuration space is presented.

1. – Introduction.

In a previous paper (1) it has been shown that it is possible to define a potential that describes the interactions of the nucleon-nucleon system by a generalization of the method of Charap and Fubini (2).

Because of the additional degrees of freedom introduced by the spin of the particles as compared with the scalar case, the system of integral equations to be solved to actually construct the various potentials necessary to describe the interaction is much more complicated than in the scalar problem.

The system of integral equations turns out to be formed by two independent sets associated with the eigenvalues 0 and 1 of T the isotopic spin operator, relating the various scattering amplitudes $T_j = t_j P_j$ and their imaginary parts. P_j represent the perturbative invariants introduced by AMATI, LEADER and VITALE (3) in their treatment of the nucleon-nucleon problem.

From the scalar functions t_j the potentials are defined. (See eq. (2.6)).

(1) F. MOREY: *Nuovo Cimento*, **24**, 585 (1962). To be referred to as I.

(2) J. M. CHARAP and S. P. FUBINI: *Nuovo Cimento*, **14**, 540 (1959); **15**, 73 (1960).

(3) D. AMATI, E. LEADER and B. VITALE: *Nuovo Cimento*, **17**, 68 (1960); **18**, 409 (1960), to be referred to as ALV I and ALV II, respectively.

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If we suppose that the various scattering amplitudes have the same dependence on the nucleon mass, it is possible to generalize Charap and Fubini's method for the construction of the potentials up to second order in a nonrelativistic approximation.

However, this assumption leads to the selection of one of the potentials as more important than the rest at low energies with the consequence that a spin orbit-spin orbit and a momentum tensor potential appear with the same weight in $1/m$ as the spin orbit and the tensor potential.

This is not in agreement with the usual phenomenological potentials introduced to describe nucleon-nucleon interactions ([4]).

It is then important to determine the real dependence on the nucleon mass of the various scalar functions.

The first part of this paper and the Appendix are devoted to determine this mass-dependence, showing that the whole potential is homogeneous in its dependence on m. This means that the system of equations to be solved cannot be split in the nonrelativistic limit and therefore, it is not possible to generalize the method of construction of the potentials mentioned earlier. This is due to the fact that then, the transition probability is not formed any longer by contributions of different order in $1/m$ coming from the various scalar functions.

The construction of the potentials must then be performed either by numerical methods or by extracting all the information from the theory of dispersion relations.

In Sections **3** and **4** the potentials are transformed to configuration space and to the nonrelativistic limit in order to obtain the usual central, spin orbit etc., types of potentials defined in terms of the potentials here introduced.

In Section 4 a method to transform to the non relativistic limit taking into account the spreading of the interacring particles in the transformation ([5]) is sketched and the resulting interaction deriving from one of the obtained potentials is presented.

2. – The definition and construction of the nucleon-nucleon potentials.

As shown in I, the definition of the potentials can be performed by considering the equivalence of the equations that determine the relationships between each of the scalar functions, coefficients of the perturbative invariants in a double dispersion-relations analysis of the scattering process, and the

([4]) R. J. N. PHILLIPS: *Rep. Progr. Phys.*, **22**, 562 (1959).
([5]) L. FOLDY and S. WOUTHUYSEN: *Phys. Rev.*, **78**, 29 (1950).

corresponding equations for the scattering amplitudes derived from the theory of potential scattering, when the latter are appropriately defined.

We will keep the notation of paper I calling $p_1 n_1$ and $p_2 n_2$ the energy momentum four-vectors for the initial and final states of the interacting nucleons. From these we define the vectors

$$(2.1)\quad P = \frac{p_1 + p_2}{2}\,;\quad N = \frac{n_1 + n_2}{2}\,;\quad \Delta = n_1 - n_2 = p_2 - p_1\,;\quad \boldsymbol{n} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \boldsymbol{\Delta} \times \boldsymbol{N}.$$

The scattering amplitude for potential scattering can be written as

$$(2.2)\qquad T = \sum_{i=1}^{5} (T_{\mathrm{D}_i} + T_{\mathrm{E}_i} P^{\tau})\,,$$

where, T_{D} and T_{E} stand for direct and exchange amplitudes, while

$$(2.3)\qquad P^{\tau} = \tfrac{1}{2}(1 + \boldsymbol{\tau}^n \cdot \boldsymbol{\tau}^p)$$

is the isotopic spin operator. Each $T_{(\mathrm{D,E})_i}$ is defined as associated with each of the perturbative invariants introduced by ALV I, that is $T_i = t_i P_i$, where the perturbative invariants are defined as

$$(2.4)\qquad \begin{cases} P_1 = 1^n 1^p\,, & P_3 = \gamma^n \cdot P \gamma^p \cdot N\,, & P_5 = \gamma_5^n \gamma_5^p\,, \\ P_2 = \gamma^n \cdot P 1^p + \gamma^p \cdot N 1^n\,, & & P_4 = \gamma^n \cdot \gamma^p\,. \end{cases}$$

Then introducing the scalars

$$(2.5)\quad s = (p_1 + n_1)^2\,;\quad t = (p_2 - p_1)^2 = (n_1 - n_2)^2\,;\quad u = (n_1 - p_2)^2 = (n_2 - p_1)^2,$$

the potentials are defined through the relations

$$(2.6)\qquad V_{\mathrm{D}_j}(t) = t_{\mathrm{D}_i}(0, u)_t - \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im} t_{\mathrm{D}_i}(\eta'^2, t)}{\eta'^2}\, \mathrm{d}\eta'^2\,.$$

The results of potential scattering can then be made to coincide with those of the theory of dispersion relations if one imposes the conditions on

$$(2.7)\qquad t_{\mathrm{D}_j}(0, u) = p_{\mathrm{D}_j}(0, u)_t\,,$$

where the p_{D_j} are the scalar functions coefficients of the perturbative invariants in double dispersion relations theory.

If we assume that the p_{D_j}'s are independent of m, the nucleon mass, or that their dependence on m is homogeneous, that is, the same for all p_{D_j}, then a generalization of the method of constructing the potentials given by CHARAP and FUBINI (2) could be devised, allowing formally the solution of the problem to second order in a nonrelativistic approximation.

However one can show that this is not the situation, at least in two pions exchange processes. These have been studied extensively by AMATI, LEADER and VITALE, under the assumptions that the $J=1$, $T=1$, π-π resonance and the $\frac{3}{2}$, $\frac{3}{2}$, π-$\mathcal{N}$ resonance are dominating. Using these results, which one can expect to be at least qualitatively correct, even if for example the 3π resonance proved to be so strong as to give a very important contribution to the process, one can write, as shown in the Appendix

$$
\left\{
\begin{aligned}
& p_1(s,t) = m^{-2}\mathscr{P}_1\left(\frac{s}{m^2},\frac{t}{m^2}\right), \\
& p_2(s,t) = m^{-3}\mathscr{P}_2\left(\frac{s}{m^2},\frac{t}{m^2}\right), \\
& p_3(s,t) = m^{-4}\mathscr{P}_1\left(\frac{s}{m^2},\frac{t}{m^2}\right), \\
& p_4(s,t) = m^{-2}\mathscr{P}_4\left(\frac{s}{m^2},\frac{t}{m^2}\right), \\
& p_5(s,t) = 0 .
\end{aligned}
\right. \tag{2.8}
$$

The potentials defined from these functions will not therefore have the same mass-dependence, but the one given by 2.8.

The nonrelativistic limit of the invariants is given by (1)

$$
\left\{
\begin{aligned}
& P_1 = 1 + \frac{i}{(E+m)^2}(\boldsymbol{\sigma}^n + \boldsymbol{\sigma}^p)\cdot\boldsymbol{n} - \frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^4}, \\
& P_2 = 2A + iC(\boldsymbol{\sigma}^n + \boldsymbol{\sigma}^p)\cdot\boldsymbol{n} + \frac{2B}{(E+m)^2}\boldsymbol{\sigma}^p\cdot\boldsymbol{n}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{n}, \\
& P_3 = A^2 - iAB(\boldsymbol{\sigma}^p + \boldsymbol{\sigma}^n)\cdot\boldsymbol{n} - B^2\boldsymbol{\sigma}^p\cdot\boldsymbol{n}\boldsymbol{\sigma}^n\cdot\boldsymbol{n}, \\
& P_4 = 1 - \frac{3i}{(E+m)^2}(\boldsymbol{\sigma}^p + \boldsymbol{\sigma}^n)\cdot\boldsymbol{n} + \\
& \qquad + \frac{1}{(E+m)^2}(\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{\Delta} - \boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\boldsymbol{\Delta}^2) - \frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^4}, \\
& P_5 = \boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\,\boldsymbol{\sigma}^p\boldsymbol{\Delta}/(E+M)^2,
\end{aligned}
\right. \tag{2.9}
$$

where

$$(2.10)\quad A = P_0\left[1 + \frac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{(E+m)^2} + \frac{2\boldsymbol{P}^2}{(E+m)P_0}\right];\quad B = \frac{P^0}{(E+m)^2} - \frac{1}{(E+m)};$$

$$C = \frac{A}{(E+m)^2} - B.$$

Therefore, the resulting potential can be written in the non relativistic limit as

$$(2.11)\quad V = (V_1 + 2AV_2 + A^2V_3 + V_4) +$$

$$+ \frac{2}{i}[-V_1/(E+m)^2 - CV_2 + ABV_3 + 3V_4/(E+m)^2]\,\boldsymbol{S}\cdot\boldsymbol{n} +$$

$$+ \left[-V_1/(E+m)^4 + \frac{2B}{(E+M)^2}V_2 - B^2V_3 - \frac{V_4}{(E+m)^4}\right]\boldsymbol{\sigma}^p\cdot\boldsymbol{n}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{n} +$$

$$+ \frac{V_4}{(E+m)^2}[\sigma^n\cdot\Delta\,\sigma^p\cdot\Delta - \boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\Delta^2].$$

Therefore, if we write the expression for the transition probability derivable from this potential or from its corresponding scattering amplitudes, it is not possible to separate from it contributions of different weight in the non relativistic limit; this then means that the method of Charap and Fubini can be generalized to the definition of potentials in the nucleon nucleon problem, but not to their construction (*). To construct the potentials, one simply has to derive all the information from the theory of dispersion relations. This can be done presently for two pions interchanged by numerical methods. The resulting potential will then give rise to the different types used in the phenomenological descriptions of scattering; to identify them we must transform to configuration space.

3. – The potentials in configuration space.

The obtained potentials are non local, therefore Schrödinger's equation reads

$$(3.1)\quad (E + \nabla^2/m)\psi(r) = \int \langle rVr'\rangle\,\psi(r')\,\mathrm{d}\boldsymbol{r}'.$$

(*) See ref. (1), Section 5.

GOTO and MACHIDA (6) have shown that one can write quite generally

$$(3.2)\qquad \langle rVr'\rangle = \int \frac{\mathrm{d}\boldsymbol{\Delta}\,\mathrm{d}\boldsymbol{P}}{(2\pi)^3} \exp\,[+\,i\boldsymbol{N}\cdot(\boldsymbol{r}-\boldsymbol{r}')] \exp\left[-\,i\frac{\boldsymbol{\Delta}}{2}(\boldsymbol{r}+\boldsymbol{r}')\right] V(\boldsymbol{\sigma},\boldsymbol{\Delta},\boldsymbol{N})\,.$$

Following a procedure similar to the one introduced by HOSHIZAKI and MACHIDA (7) one can express

$$(3.3)\qquad \langle rVr'\rangle = \int \frac{\mathrm{d}\boldsymbol{\Delta}}{(2\pi)^{\frac{3}{2}}} \exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]\, V\left(\boldsymbol{\sigma},\boldsymbol{\Delta},-\,i\,\frac{\partial}{\partial(\boldsymbol{r}-\boldsymbol{r}')}\right)\delta(\boldsymbol{r}-\boldsymbol{r}')\,,$$

thus obtaining

$$(3.4)\qquad \int \langle rVr'\rangle\,\varphi(r')\,\mathrm{d}\boldsymbol{r}' = \int \frac{\mathrm{d}\boldsymbol{\Delta}}{(2\pi)^{\frac{3}{2}}} \exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]\, V\left(\boldsymbol{\sigma},\boldsymbol{\Delta},-\,i\,\frac{\vec{\partial}}{\partial r}\right)\varphi(r) =$$

$$= V\left(\boldsymbol{\sigma},\boldsymbol{r},-\,i\,\frac{\vec{\partial}}{\partial r}\right)\varphi(r)\,.$$

We notice that $\boldsymbol{N}$ is replaced everywhere by $-i\vec{\partial}/\partial r$, which corresponds to p, the relative momentum of the particles in the quantum mechanical formalism (*).

We can then proceed to the transformation of the potentials V_i obtaining

$$(3.5)\qquad \int \frac{V_i}{(2\pi)^{\frac{3}{2}}}\,\boldsymbol{S}\cdot\boldsymbol{\Delta}\times\boldsymbol{N}\exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{\Delta} = -\,\frac{1}{i}\frac{1}{r}\frac{\partial}{\partial r}\,V_i(r)\boldsymbol{S}\cdot\boldsymbol{r}\times\boldsymbol{p}\,,$$

$$(3.6)\qquad \int \frac{V_i}{(2\pi)^{\frac{3}{2}}}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\times\boldsymbol{N}\boldsymbol{\sigma}^p\cdot\boldsymbol{\Delta}\times\boldsymbol{N}\exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{\Delta} =$$

$$= \frac{1}{r}\frac{\partial V_i}{\partial r}\,[\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{p} - \boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\boldsymbol{p}^2] - \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_i\boldsymbol{\sigma}^n\cdot\boldsymbol{L}\boldsymbol{\sigma}^p\cdot\boldsymbol{L}\,,$$

$$(3.7)\qquad \int \frac{V_i}{(2\pi)^{\frac{3}{2}}}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\boldsymbol{\Delta}^2\exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{\Delta} = -\,\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\left[\frac{1}{r}\frac{\partial}{\partial r}\,V_i + r^2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_i\right],$$

$$(3.8)\qquad \int \frac{V_i}{(2\pi)^{\frac{3}{2}}}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\boldsymbol{\sigma}^p\cdot\boldsymbol{\Delta}\exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{\Delta} = -\,\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\,\frac{1}{r}\frac{\partial}{\partial r}\,V_i - \boldsymbol{\sigma}^n\cdot\boldsymbol{r}\boldsymbol{\sigma}^p\cdot\boldsymbol{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_i\,.$$

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(6) J. GOTO and S. MACHIDA: *Progr. Theor. Phys.*, **25**, 64 (1960).

(7) H. HOSHIZAKI and S. MACHIDA: *Progr. Theor. Phys.*, **24**, 1317 (1960).

(*) Notice however that if one is to perform a perturbative type of calculation the operator $\boldsymbol{N}$ is to be identified with $-(i/2)(\overleftarrow{\partial}/\partial r + \vec{\partial}/\partial r)$ (see ref. (7)). Notice also the difference in sign of the exponential $\exp\,[-\,i\boldsymbol{\Delta}\cdot\boldsymbol{r}]$ in eq. (3.3) as compared with the corresponding sign in eq. (A.2) of ref. (7).

Therefore

$$V = V_c(\boldsymbol{r}) + V_{so}\boldsymbol{S}\cdot\boldsymbol{L} + V_T(\boldsymbol{\sigma}^n\cdot\boldsymbol{r}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{r} - \boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p r^2) + V_{soso}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{L}\boldsymbol{\sigma}^p\cdot\boldsymbol{L} + \\ + V_{T_p}(\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{p} - \boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p p^2)\,, \tag{3.9}$$

where V_c, V_{so}, etc., stand for central, spin orbit, tensor, spin orbit-spin orbit and momentum tensor potentials. In the nonrelativistic limit of the coefficients A, B, and C, namely

$$A = m\,; \qquad B = -\tfrac{1}{4}m\,; \qquad C = \tfrac{1}{2}m \tag{3.10}$$

we can then write

$$\begin{cases} V_c = V_1 + 2mV_2 + m^2V_3 + V_4 = V_{c_1} + 2V_{c_2} + V_{c_3} + V_{c_4} \\ V_{so} = -\dfrac{1}{2m^2}\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)(V_{c_1} + 2V_{c_2} + V_{c_3} - 3V_{c_4})\,, \\ V_T = -\dfrac{1}{4m^2}\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)^2 V_{c_4}\,, \\ V_{soso} = \dfrac{1}{16m^4}\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)^2(V_{c_1} + 2V_{c_2} + V_{c_3} + V_{c_4})\,, \\ V_{T_p} = -\dfrac{1}{16m^4}\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)(V_{c_1} + 2V_{c_2} + V_{c_3} + V_{c_4})\,, \end{cases} \tag{3.11}$$

which qualitatively agree with the potentials introduced in phenomenological analysis.

This potential contains however only the first order effects of each of the different types of interactions; it shows explicitely the nature and the origin of the various low energy potentials as referred to the V_i defined at arbitrary energies sufficiently below threshold. It is however not consistent; for consistency it must contain all terms up to a given power in p/m, including all the terms that account for the spreading of the interacting particles when a low energy limit is taken ([5]). To disregard the terms that are missing in eqs. (3.9), (3.11) means to disregard these very important effects. To obtain the complete nonrelativistic limit of a given Hamiltonian we may proceed in two ways. We can perform a series of Foldy-Wouthuysen transformations until a Hamiltonian free from odd operators of the desired order in p/m is

obtained and from it proceed to separate large and small components, or we can follow Rosenfeld's approach using appropriate projection operators ([8]).

4. – Transformation of high-energy potentials to the nonrelativistic limit.

The results obtained by both methods are similar, but the second is algebraically simpler. The behaviour of two fermions interacting through a given potential V is described by the Hamiltonian

$$H = (\varrho_1^n \boldsymbol{\sigma}^n - \varrho_1^p \boldsymbol{\sigma}^p)\cdot \boldsymbol{p} + \tfrac{1}{2}(\varrho_3^n + \varrho_3^p)2m + V, \tag{4.1}$$

where $\boldsymbol{\sigma}^n$ and $\boldsymbol{\sigma}^p$ represent the spin operators for particles n and p; $\boldsymbol{p}$ is the relative momentum of the system and ϱ_1 and ϱ_3 are two dicotomic variables associated with particles n and p, represented by two-by-two matrices. If we choose the representation in which ϱ_3 is diagonal, it has the eigenvalues ± 1 associated with the two components of the eigenfunctions $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$. In eq. (4.1) the operator $\varrho_3 = \frac{1}{2}(\varrho_3^n + \varrho_3^p)$ acts; it will have as eigenfunctions as usual the functions

$$^3\varrho_1 = u^n u^p; \quad ^3\varrho_0 = \frac{u^n v^p + u^p v^n}{\sqrt{2}}; \quad ^1\varrho_0 = \frac{u^n v^p - v^n v^p}{\sqrt{2}}; \quad ^3\varrho_{-1} = v^n v^p, \tag{4.2}$$

belonging to the eigenvalues 1, 0 and -1 of ϱ_3. To perform the separation of large and small components of the wave function that can be written as

$$\psi = {}^3f_1\, {}^3\varrho_1 + ({}^3f_0\, {}^3\varrho_0 + {}^1f_0\, {}^1\varrho_0) + {}^3f_{-1}\, {}^3\varrho_{-1}, \tag{4.3}$$

where ${}^3f_1$ represents the large component of the two particle wave function and ${}^3f_0$, ${}^1f_0$, and ${}^3f_{-1}$, the components of order p/m and $(p/m)^2$ respectively, we introduce the operators $P_\pm = \frac{1}{2}(1 \pm \varrho_3)$ and $\varrho_\pm = \varrho_1 P_\mp$ with the properties

$$\begin{aligned} &P_+ u = u; \quad P_+ v = 0; \quad P_- u = 0; \quad P_- v = v; \\ &\varrho_+ u = 0; \quad \varrho_+ v = u; \quad \varrho_- u = v; \quad \varrho_- v = 0, \end{aligned} \tag{4.4}$$

multiplying (4.1) by the operators $P_+^n P_+^p$, $P_+^n P_-^p$, $P_-^n P_+^p$ and $P_-^n P_-^p$ in succession and then adding the results. If we specialize for instance to potential V_3, with $V_{111} = V_{c_3} N_0^2$ we obtain as the acting potential

$$V = V_{111} + V_{111}/N_0(\varrho_1^p \boldsymbol{\sigma}^p - \varrho_1^n \boldsymbol{\sigma}^n)\cdot \boldsymbol{p} - V_{111}/N_0^2\,(\varrho_1^n \varrho_1^p\, \boldsymbol{\sigma}^p \cdot \boldsymbol{p}\, \boldsymbol{\sigma}^n \cdot \boldsymbol{p}) \tag{4.5}$$

([8]) L. ROSENFELD: *Nuclear Forces* (Amsterdam, 1948).

and then the set of equations

$$
(4.6)\quad \begin{cases}
(1-V_{111}/N_0)(\alpha_-\,{}^3f_0-\alpha_+\,{}^1f_0)-(V_{111}/N_0^2)\,\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{p}\,{}^3f_{-1}+V_{111}\,{}^3f_1=\varepsilon\,{}^3f_1\,,\\
(1-V_{111}/N_0)(\alpha_-\,{}^3f_0+\alpha_+\,{}^1f_0)-(V_{111}/N_0^2)\,\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{p}\,{}^3f_1+(E+2m-V_{111})\,{}^3f_{-1}=0\,,\\
(1-V_{111}/N_0)\alpha_-({}^3f_1+{}^3f_{-1})-(V_{111}/N_0^2)\,\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{p}\,{}^3f_0+(V_{111}-E)\,{}^3f_0=0\,,\\
-(1-V_{111}/N_0)\alpha_+({}^3f_1-{}^3f_{-1})+(V_{111}/N_0^2)\,\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\,\boldsymbol{\sigma}^p\cdot\boldsymbol{p}\,{}^1f_0+(V_{111}-E)\,{}^1f_0=0\,,
\end{cases}
$$

where we have introduced

$$
(4.7)\qquad \alpha_\pm=\frac{1}{\sqrt{2}}(\boldsymbol{\sigma}^n\pm\boldsymbol{\sigma}^p)\cdot\boldsymbol{p}\qquad\text{and}\qquad \varepsilon=E-2m\,.
$$

To derive eqs. (4.6), use has been made of the orthogonality of the functions ${}^3\varrho_1$, ${}^3\varrho_0$, ${}^3\varrho_{-1}$, and ${}^1\varrho_0$.

From this set we can obtain an equation in terms of the large component of the wave function exact to any order in p/m. To the fourth order we get for potential V_3

$$
\begin{aligned}
(4.8)\qquad V=&\left\{1-\frac{2\boldsymbol{p}^2}{m^2}+\frac{V_{111}\boldsymbol{p}^2}{m^3}+\frac{3\boldsymbol{p}^4}{m^4}+\frac{i}{2m^2}\boldsymbol{r}\cdot\boldsymbol{p}\frac{1}{r}\frac{\partial}{\partial r}-\frac{i}{2m^4}\boldsymbol{r}\cdot\boldsymbol{p}V_{111}^2\frac{1}{r}\frac{\partial}{\partial r}+\right.\\
&+\frac{1}{16m^4}\left[(-6i\boldsymbol{r}\cdot\boldsymbol{p}\boldsymbol{p}^2+9\boldsymbol{p}^2)\frac{1}{r}\frac{\partial}{\partial r}+(2r^2\boldsymbol{p}^2+3(\boldsymbol{r}\cdot\boldsymbol{p})^2+10i\boldsymbol{r}\cdot\boldsymbol{p})\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2-\right.\\
&\left.\left.-2ir^2\boldsymbol{r}\cdot\boldsymbol{p}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\right]\right\}V_{111}-\left\{\left(\frac{1}{2m^2}-\frac{i}{2m^4}V_{111}^2\boldsymbol{r}\cdot\boldsymbol{p}\right)\frac{1}{r}\frac{\partial}{\partial r}+\right.\\
&\left.+\frac{1}{16m^4}\left[10p^2\frac{1}{r}\frac{\partial}{\partial r}+(2i\boldsymbol{r}\cdot\boldsymbol{p}+10)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\right]\right\}V_{111}\boldsymbol{S}\cdot\boldsymbol{L}-\\
&-\frac{1}{16m^4}\frac{1}{r}\frac{\partial}{\partial r}V_{111}(\boldsymbol{\sigma}^n\cdot\boldsymbol{p}\boldsymbol{\sigma}^p\cdot\boldsymbol{p}-\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\boldsymbol{p}^2)+\frac{1}{16m^4}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2V\boldsymbol{\sigma}^n\cdot\boldsymbol{L}\boldsymbol{\sigma}^p\cdot\boldsymbol{L}\,.
\end{aligned}
$$

The necessary calculations are lengthy but not too awkward, and are published elsewhere. In this equation the operator $\boldsymbol{p}$ acts only on the wave function, the derivatives only on the potential.

Similar equations are obtained for the rest of the potentials by the same method. When all potentials are supposed to act simultaneously, as in fact they act, new terms appear due to the interactions among them.

It is therefore clear that to describe nucleon-nucleon scattering, it is not enough to consider the existence of central, spin-orbit, etc., types of interaction potentials of the direct and exchange type. No potential can be built

from a non relativistic limit of a theory, whether dispersion relations, field theory or other containing only simple type of potentials as central, spin-orbit, etc. All the rest of terms accounting for the spreading must be also present; there can be no hope to fitting experimental data for any appreciable range of energies with these terms missing. Therefore, the concept of potential itself ceases to be a helpful one if one wants to go beyond second-order nonrelativistic approximations, and even then some nonstatic terms must be kept in mind such as the term $-2V_{111}\boldsymbol{p}^2/m^2$ in eq. (4.7). Even so, the quantitative determination of potentials V_1 to V_4 is of interest for they will allow a check on the theory of dispersion relations as well as on the correctness of the approximations made in their calculation, such as dominating character of $\pi\pi$ and $\pi\mathcal{N}$ resonances.

APPENDIX

We will use the following equations from ALV I

$$p_i(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho_i(s,t')}{t'-t}\,\mathrm{d}t' \,, \tag{A.1}$$

where we have simplified their eq. (3.7) to fit our purpose, namely the mass-dependence of the functions p_i.

Also we need

$$\sigma_A^{\pm} = -2\bar{}_A\sigma(s,t) = \frac{8\pi}{3}\left[\frac{\sqrt{s}+m}{E+m}\,3\left(1+\frac{t}{2k^2}\right)+\frac{\sqrt{s}-m}{E-m}\right]\frac{\sin^2\delta_{33}(s)}{k}\,, \tag{A.2}$$

$$\sigma_B^{+} = -2\sigma_B^{+} = \frac{8\pi}{3}\left[\frac{3}{E+m}\left(1+\frac{t}{2k^2}\right)-\frac{1}{E-m}\right]\frac{\sin^2\delta_{33}(s)}{k}\,;\ \bar{\sigma}_B^{\pm} = \sigma_B^{\pm}+\pi g^2\delta(s-m^2), \tag{A.3}$$

with

$$k^2 = \frac{\frac{1}{4}(s-m^2-\mu^2)^2-\mu^2m^2}{s}\,; \qquad E = \frac{s+m^2-\mu^2}{2\sqrt{s}} \tag{A.4}$$

One can easily see that

$$k^2 = m^2\frac{\frac{1}{4}(s/m^2-1-\mu^2/m^2)^2-\mu^2/m^2}{s/m^2}\,, \tag{A.5}$$

and therefore

$$\sigma_A^{\pm} = \frac{1}{m}\varphi^{\pm}(s/m^2,\, t/m^2)\,; \qquad \sigma_B^{\pm} = \frac{1}{m^2}\vartheta^{\pm}(s/m^2,\, t/m^2)\,. \tag{A.6}$$

Therefore from expressions (3.40) and (3.39) in ALV I

$$\text{(A.7)}\quad \begin{cases} K(s', s'', t, x) = \{t(x-[\sqrt{s'}+\sqrt{s''}]^2)(x-[\sqrt{s'}-\sqrt{s''}]^2) - 4m^2(s'-s'')^2 - \\ \qquad\qquad - 4x[s's''+(\mu^2-m^2)^2-(\mu^2+m^2)(s'+s'')+\mu^2 x]\}^{-\frac{1}{2}}, \\ \therefore K(s', s'', t, x) = \dfrac{1}{m^3} K(s'/m^2, s''/m^2, t/m^2, x/m^2), \end{cases}$$

and with $\zeta = s/m^2$

$$\text{(A.8)}\quad \begin{cases} \varkappa_a^{\pm} = \sqrt{t}\iint \mathrm{d}s'\mathrm{d}s''\sigma_A^{\pm}(s', t)\sigma_A^{\pm}(s'', t)K(s', s'', t, x) = \\ \qquad\qquad = m\sqrt{t/m^2}\iint \mathrm{d}\zeta'\mathrm{d}\zeta''\dfrac{1}{m}\varphi^{\pm}(\zeta', t/m^2)\varphi^{\pm}(\zeta'', t/m^2)K, \\ \therefore \varkappa_a = m^0 F_1(t/m^2, x/m^2). \end{cases}$$

And similarly with

$$\text{(A.9)}\quad \begin{cases} Z(s', s'', t) = m^2[\zeta'+\zeta''-2(\mu^2/m^2+1)+t/m^2], \\ Y(s', s'') = m^2(\zeta'-\zeta''). \end{cases}$$

In the same way one obtains

$$\text{(A.10)}\quad \begin{cases} \varkappa_b = \dfrac{m}{x-4m^2+t}F_2(t/m^2, x/m^2) + \dfrac{m}{x}F_3(t/m^2, x/m^2), \\ \varkappa_c = \dfrac{m}{x-4m^2+t}F_2 - \dfrac{m}{x}F_3, \\ \varkappa_d = m^4\dfrac{t-4m^2}{x^2(x-4m^2+t)^2}F_4 + \dfrac{m^2}{(x-4m^2+t)^2}F_5 + \dfrac{m^2}{x^2}F_6, \\ \varkappa_e = \dfrac{m^2}{(x-4m^2+t)^2}F_5 - \dfrac{m^2}{x^2}F_6 - m^4\dfrac{2x-4m^2+t}{x^2(x-4m^2+t)^2}F_4. \\ \varkappa_f = \dfrac{m^4}{x(x-4m^2+t)}F_4. \end{cases}$$

Therefore, from expressions (3.43), ALV I, for instance for

$$\text{(A.11)}\quad \varrho_1^{\pm}(s, t) = \frac{1}{2(2\pi)^2 t}\left[\pm\frac{1}{\pi}\int_{(2m)^2}\frac{\varkappa_a^{\pm}+2m\varkappa_c^{\pm}+m^2\varkappa_e}{x-s}\mathrm{d}x + \frac{\pi}{4}\left(t\left(\frac{t}{4}-\mu^2\right)\right)^{\frac{1}{2}}\cdot \left(\frac{|\lambda_0^+|^2-\alpha_0^{+2}}{(-s/6)(t/4-\mu^2)(|\lambda_1^-|^2-\alpha_1^{-2})}\right)\right].$$

Using formulae (4.11) and (4.12) in ALV II (from BOWCOCK, COTTINGHAM and LURIÉ [9]) one can show that the second term in eq. (A.11) has the same mass-dependence than the first one. The same is valid for all ϱ_i (i from 1 to 4). One then obtains

$$\text{(A.12)} \qquad \begin{cases} \varrho_1 = m^{-2}\mathscr{R}_1(\zeta, t/m^2)\,, & \varrho_3 = m^{-4}\mathscr{R}_3\,, \\ \varrho_2 = m^{-3}\mathscr{R}_2\,, & \varrho_4 = m^{-2}\mathscr{R}_4\,. \end{cases}$$

Which is by (A.1), the mass weight of the scalar functions p_i and hence of the potentials.

(9) J. BOWCOCK, W. COTTINGHAM and D. LURIÉ, *Nuovo Cimento*, **16**, 918, (1960).

RIASSUNTO (*)

Si mostra che i potenziali che possono essere definiti per il problema nucleone-nucleone della teoria delle relazioni di doppia dispersione, non si possono costruire con una generalizzazione del metodo di Charap e Fubini. Si presenta una espressione esplicita per l'identificazione dei potenziali nel limite non relativistico nello spazio delle configurazioni.

(*) *Traduzione a cura della Redazione.*