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# Example of a Gauge Transformation which Cannot Be Continuously Lifted to the Lie Algebra.

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The habit of writing every element of a Lie group $G$ as exponential of some element of the Lie algebra of $G$, which is usual in physical texts (1), is of course perfectly legitimate in the usual case ($G$ compact and connected). But it can be somewhat misleading as it induces to think that every continuous gauge transformation

$$\Omega\colon X \to G \tag{1}$$

($X$ some subset of space-time, $G$ some compact connected Lie group) can be written in the form

$$\Omega(x) = \exp[\Lambda(x)] \tag{2}$$

with a continuous $\Lambda$ (taking its values in the Lie algebra $\mathbf{G}$ of $G$).

Even if, in many circumstances, this inaccuracy has no bad consequences, greater care is necessary when topological considerations are of relevance. To see that the restriction to gauge transformations of the form (2) (with *continuous* $\Lambda$) is really an undesirable limitation it suffices to note that every such $\Omega$ can be continuously deformed to a constant one, whatever the $x$ may be: $\Omega_t(x) = \exp[t\Lambda(x)]$, $0 \leqslant t \leqslant 1$.

The situation is not too bad when $G = U_1$. For instance, in writing

$$\Omega(x) = \exp[i \arg x] \qquad (x \in \mathbf{R}^2 - \{0\}),$$

even if arg cannot be chosen as a continuous function from $\mathbf{R}^2 - \{0\}$ to $\mathbf{R}$, its discontinuities can be shifted away from any point chosen in advance.

In the non-Abelian case, the problem can be worse than that, and thus it may be of interest to give a simple example of an $\Omega\colon \mathbf{R}^4 \to SU_2$ such that if $\Omega(x) = \exp[\Lambda(x)]$, $\Lambda$ is necessarily discontinuous in the neighbourhood of certain fixed points.

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(1) A. ABERS and B. W. LEE: *Phys. Rev. C*, **9**, 1 (1973); W. MARCIANO and H. PAGELS: *Phys. Rev. C*, **36**, 137 (1978); J. C. TAYLOR: *Gauge Theories of Weak Interactions* (Cambridge, Mass., 1979).

We recall that the exponential mapping exp: $\underline{SU}_2 \to SU_2$ is given by

$$\exp[iA] = (\cos r)I + \frac{\sin r}{r} iA , \tag{3}$$

where

$$A = \begin{pmatrix} y_1 & y_2 - iy_3 \\ y_2 + iy_3 & -y_1 \end{pmatrix} = y_1\sigma_z + y_2\sigma_x + y_3\sigma_y , \qquad r = |A| = \sqrt{y_1^2 + y_2^2 + y_3^2} \tag{4}$$

and is one-to-one and regular from the ball $|A| < \pi$ onto $SU_2 - \{-I\}$. If we write $U \in SU_2$ as

$$U = \begin{pmatrix} z_1 + iz_2 & z_3 + iz_4 \\ -z_3 + iz_4 & z_1 - iz_2 \end{pmatrix} \qquad (z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1) , \tag{5}$$

the exp mapping, in terms of $y$ and $z$ is

$$(y_1, y_2, y_3) \to (z_1, z_2, z_3, z_4) = \left(\cos r, \frac{\sin r}{r} y_1, \frac{\sin r}{r} y_3, \frac{\sin r}{r} y_2\right) \tag{6}$$

and, from $SU_2 - \{-I\}$ to $\{|A| < \pi\}$ has the unique inverse $L$

$$L(z_1, z_2, z_3, z_4) = \left(\frac{\arccos z_1}{+\sqrt{1 - z_1^2}} z_2, \frac{\arccos z_1}{+\sqrt{1 - z_1^2}} z_4, \frac{\arccos z_1}{+\sqrt{1 - z_1^2}} z_3\right) \tag{7}$$

$(0 \leqslant \arccos z_1 < \pi)$.

Let $C$ be the cylinder in $\mathbf{R}^4$

$$C = \{x \in \mathbf{R}^4;\ x_1^2 + x_2^2 \leqslant 1\}$$

and let $\Omega$: $C \to SU_2$ be given by

$$\Omega(x) = \begin{pmatrix} x_1 + ix_2 & +\sqrt{1 - (x_1^2 + x_2^2)} \\ -\sqrt{1 - (x_1^2 + x_2^2)} & x_1 - ix_2 \end{pmatrix} . \tag{8}$$

(The points of $C$ which are mapped by $\Omega$ to $-I$ are those of the subset $S = \{x \in C;\ x_1 = -1,\ x_2 = 0\}$). If we had a continuous $A$: $C \to \underline{SU}_2$ such that $\Omega(x) = \exp[A(x)]$, and with $A(1, 0, 0, 0) = 0$, for $x \in C - S$ we should necessarily have

$$A(x) = L(\Omega(x)) = \left(\frac{\arccos x_1}{\sqrt{1 - x_1^2}} x_2, 0, \frac{\arccos x_1}{\sqrt{1 - x_1^2}} \sqrt{1 - (x_1^2 + x_2^2)}\right) . \tag{9}$$

But this $A$ cannot be continuously extended to the points of $S$: for instance, if $x$ converges to $(-1, 0, 0, 0)$ on the line $x_2 = x_3 = x_4 = 0$, $A(x)$ converges to $(0, 0, \pi)$, while if the convergence to $(-1, 0, 0, 0)$ is along the circumference $x_1^2 + x_2^2 = 1$, $x_3 = x_4 = 0$, $A(x)$ converges to $(\pi, 0, 0)$.

We conclude with a few remarks.

*a*) In the example above, $\Omega(1, 0, 0, 0) = I$ and therefore $A(1, 0, 0, 0)$ must be of norm $r = 2k\pi$. We have chosen $k = 0$, but nothing is to be gained by choosing $k \neq 0$; rather, the problem in extending $A$ would arise *also* in the neighbourhood of $(1, 0, 0, 0)$.

*b*) Using some continuous $F: \mathbf{R}^4 \to C$ with $F(x) = x$ for $x \in C$, the $\Omega$ above can be extended to an $\tilde{\Omega} = \Omega \circ F: \mathbf{R}^4 \to SU_2$, which, like $\Omega$, cannot be lifted to $\underline{SU}_2$.

*c*) The restriction of the $\Omega$ (or $\tilde{\Omega}$) given above to *any* $X \subseteq \mathbf{R}^4$ can be continuously deformed to a constant (because the image $\Omega(C)$ can be deformed to a point in $SU_2$). Therefore, not even this strong limitation over a gauge transformation can ensure the possibility of its lifting to the Lie algebra.

*d*) The usual formulae for gauge transformations necessary to prove the invariance of a Lagrangian of the form

$$\mathscr{L} = -\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{10}$$

are valid, and can be proved in elementary form ([2]), for general $\Omega$ (eq. (1)) not necessarily of the form of eq. (2).

This means that if the spinor field transforms according to $\psi \to \Omega\psi$, the compensating field appearing in $D_\mu = \gamma_\mu - A_\mu$ must transform as

$$A_\mu \to \Omega A \Omega^{-1} + (\partial_\mu \Omega)\Omega^{-1} \tag{11}$$

so that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - |A_\mu, A_\nu| \to \Omega F_{\mu\nu}\Omega^{-1} \tag{12}$$

to ensure the invariance of $\mathscr{L}$ in eq. (10).

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([2]) W. DRECHSLER and M. E. MAYER: *Fiber Bundle Techniques in Gauge Theories, Springer Lecture Notes in Physics*, No. 67 (1977).

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