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" ON A COMPLETE SET OF FUNCTIONS ASSOCIATED WITH A PECULIAR EIGENVALUE PROBLEM "

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ABSTRACT: Consideration of the eigenvalue problem associated with a particle in the potential $V(x) = F|x|$ leads to a complete set of functions in the interval $(-\infty, \infty)$ derived from the Airy function. Completeness and orthogonality relations give properties for the Airy function not previously noted in the literature.-

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The solutions of the Schrödinger equation for a particle in a given potential, provide us with complete sets of functions. Among the one-dimensional problems most commonly dealt with in the courses on Quantum Mechanics the simplest is certainly the harmonic oscillator, which has a discrete spectrum of equidistant levels, and whose eigenfunctions are related to the well known Hermite polynomials.

The problem presented here, of a particle in the potential $V(x) = F|x|$ provides another example which could well be included in any elementary course.

Since the corresponding classical motion is bounded, the energy spectrum is discrete[(1) (2)]. The eigenfunctions can all be generated from a single function, the Airy function. From the orthogonality and completeness relations we stablish some properties of the Airy functions which, to the best of our knowledge, had not been previously noted in the literature.

The Schrödinger equation can be written

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - F|x|)\psi = 0 \tag{1}$$

Introducing the characteristic lenght $x_o = \left(\frac{\hbar^2}{2mF}\right)^{1/3}$,

we can use the dimensionless variables .

$$\xi = \frac{x}{x_o} \qquad\qquad \xi_o = \frac{E}{Fx_o} \tag{2}$$

and write Eq(1) for $\xi > 0$, in the form

$$\frac{d^2\psi}{d\xi^2} - (\xi - \xi_o)\psi = 0 \tag{3}$$

The solution of (3) which is regular for $\xi \to \infty$ is the Airy function[2] Ai (ξ) which has the integral representation

$$\mathrm{Ai}\ (\xi) = \frac{1}{\Pi} \int_0^\infty \cos\left(\frac{1}{3} t^3 + \xi t \right) dt \qquad (4)$$

and the asymptotic behaviour for $\xi \to \infty$

$$\mathrm{Ai}\ (\xi) = \frac{1}{2} \Pi^{-1/2} \xi^{-1/4} e^{-z} \sum_0^\infty (-)^k c_k z^{-k} \qquad (5)$$

where $z = \frac{2}{3} \xi^{3/2}$

and

$$c_o = 1 \quad , \quad c_k = \frac{\Gamma\ (3k + 1/2)}{54^k\ k!\ \Gamma\ (k+1/2)} \qquad (6)$$

The general form of Ai (ξ) is given in Fig. 1

Insert Fig. 1 here

To determine the energy eigenvalues, we notice that since the potential is even, the eigenfunctions must be either even or odd[1]. We can obtain them by choosing ξ_o such that the point $\xi = o$ coincides with either a zero of the derivative of the Airy function or with a zero of the Airy function itself.

For $\xi < o$, the differential equation is

$$\frac{d^2 \psi}{d\xi^2} - (- \xi - \xi_o) \psi = 0 \qquad (7)$$

To obtain a solution valid in the whole interval $(- \infty - \infty)$ we simply continue the portion of the Airy function which we had for $\xi > o$, with the condition that the whole function be even or odd, respectively.

In this form we see that he eigenfunctions are given by

$$\psi_n(\xi) = N_n \begin{cases} Ai(\xi - \xi_n), & \xi > o \\ (-)^n Ai(-\xi - \xi_n), & \xi < o \end{cases} \tag{8}$$

where N_n is a normalization constant and ξ_n are alternatively roots of $A'i(-\xi)$ and of $Ai(-\xi)$. The first few values of ξ_n are given in Table 1[3]

TABLE 1

n	ξ_n	$Ai(-\xi_n)$	$Ai(-\xi_n)$
1	1.01879	0.53566	0
2	2.33811	0	+ 0.70121
3	3.24820	- 0.41902	0
4	4.08795	0	- 0.80311
5	4.82019	0.38041	0
6	5.52056	0	0.86520
7	6.16331	- 0.35791	0
8	6.78671	0	- 0.91085

The energy eigenvalues are given by

$$E_n = Fx_o \cdot \xi_n \quad . \tag{9}$$

For $n \to \infty$ we can use the asymptotic form for the zeros, i.e.

$$\begin{aligned} \xi_{2p} &= g\,(3\pi\,(p - 3/8)\,) \\ \xi_{2p+1} &= f\,(3\pi\,(p - 3/8)\,) \end{aligned} \tag{10}$$

where the first few terms for $f(x)$ and $g(x)$ are given by[3]

$$\begin{aligned} f\,(x) &= x^{2/3} \left[1 + \frac{5}{48}\, x^{-2} - \frac{5}{36}\, x^{-4} + \ldots \right] \\ g\,(x) &= x^{2/3} \left[1 - \frac{7}{48}\, x^{-2} + \frac{35}{288}\, x^{-4} + \ldots \right] \end{aligned} \tag{11}$$

The normalization constant can be determined be imposing the condition

$$\int_{-\infty}^{\infty} d\xi \quad \psi_n\,(\xi)^2 = 1 \tag{12}$$

which gives us

$$2N_n^2 \int_{-\xi_n}^{\infty} d\xi \quad Ai^2\,(\xi) = 1 \tag{13}$$

The integral appearing in the normalization condition can be easily done by partial integration and use of the differential equation for the Airy function. It follows then that

$$\int d\xi \quad Ai^2\,(\xi) = \xi\, Ai^2\,(\xi) - Ai'^2\,(\xi) \tag{14}$$

and therefore

$$N_n = \left\{ 2 \left[\xi_n\, Ai^2\,(-\xi_n) + Ai'^2\,(-\xi n) \right] \right\}^{1/2} \tag{15}$$

or equivalently

$$N_{2p} = \frac{1}{\sqrt{2\,\xi_{2p}}} \qquad \frac{1}{Ai\,(-\,\xi_{2p})}$$

and

$$N_{2p+1} = \frac{1}{\sqrt{2}} \qquad \frac{1}{A'i\,(-\,\xi_{2p+1})}$$

The complete set of functions thus defined must satisfy the orthogonality and completeness relations

$$\int_{-\infty}^{\infty} d\xi \;\; \psi_n\,(\xi) \quad \psi_{n'} \quad (\xi) \quad = \delta_{nn'}$$

$$\sum_n \;\; \psi_n\,(\xi') \quad \psi_n\,(\xi) \;\; = \;\; \delta\,(\xi \;-\; \xi') \qquad ,$$

which imply the following relations for the Airy function

$$\left[1 + (-)^{n+n'} \right] \int_0^{\infty} d\xi \;\; Ai\,(\xi \;-\; \xi_n) \;\; Ai\,(\xi \;-\; \xi_{n'}) = \frac{1}{N_n\, N_{n'}} \quad \delta_{nn'}$$

and

$$\sum_n \;\; N_n^2 \quad Ai\,(\xi \;-\; \xi_n) \;\; Ai\,(\xi' \;-\; \xi_n) \;=\; \delta\,(\xi \;-\; \xi')$$

Having obtained the eigenfunctions and eigenvalues it is instructive to consider some physical questions connected with the problem at hand.

One such question is to determine which portion of the total energy E_n comes from the potential energy. We recall that in the harmonic oscillator the energy is equally divided between kinetic and potential energy.

For the potential energy in an stationary state we have

$$< V_n > \;=\; F\, x_o \quad \int_{\infty}^{\infty} d\xi \;\; \xi \;\; |\; \psi_n\,(\xi)\; |^2$$

$$= \;\; N_n^2 \quad \int_{-\xi}^{\infty} d\xi \quad (\xi \;+\; \xi_n)\; Ai^2 \quad (\xi)$$

To evaluate this integral we need the following integral of the Airy function

$$\int d_\xi \; Ai^2(\xi) \cdot \xi = \frac{1}{3}\{ \xi^2 \; Ai^2(\xi) - \xi Ai'^2(\xi) + Ai(\xi) + Ai'(\xi) \} \quad (20)$$

Using (14) and (19) we obtain

$$<V>_n = \frac{4}{3} F x_o N_n^2 \{ \xi_n^2 \; Ai^2(-\xi_n) + \xi_n Ai'^2(-\xi_n) \}$$

$$= \frac{2}{3} E_n$$

We thus see that in this problem the energy of the stationary states is made up of two thirds of potential energy and one third of kinetic energy.

The next question we want to consider is the mean square deviation of the coordinate since this gives us information on the spatial distribution of the probability. We have

$$< x^2 > = x_o^2 \int_{\infty}^{\infty} d\xi \circ \xi^2 \; | \psi_n(\xi) |^2 \quad (22)$$

$$= 2x_o^2 \; N_n^2 \int_{-\xi_n}^{\infty} d\xi \; (\xi + \xi_n)^2 \; Ai^2(\xi)$$

Using now the result

$$\int \xi^2 \; Ai^2(\xi) \, d\xi = \frac{1}{5}\{ \xi^3 \; Ai^2(\xi) - \xi^2 \; Ai'^2(\xi) + 2\xi Ai(\xi) Ai'(\xi) - Ai^2(\xi) \} \quad (23)$$

we get

$$<x^2>_n = x_o^2 \; \{ \frac{8}{15} \; \xi_n^2 + \frac{1}{5} \; \frac{Ai^2(-\xi_n)}{\xi_n \; Ai^2(-\xi_n) + Ai'(-\xi_n)} \}$$

or explicitly for even and odd n

$$<x^2>_{2p} = x_o^2 \{ \frac{8}{15} \; \xi_{2p}^2 + \frac{1}{5\xi_{2p}} \} \quad (24.a)$$

and

$$<x^2>_{2p+1} = x_o^2 \; \frac{8}{15} \; {\xi_{2p+1}}^2 \quad (24.b)$$

For the uncertainty in the ground state we have

$$\Delta x^2 = \frac{1}{5\xi_o} \left[1 + \frac{8}{3} \xi_o^3 \right] x_o^2$$

It is interesting to look at the uncertainty product $\Delta x \Delta p$ for the ground state. Using the fact that the mean kinetic energy satisfies, accordin to (21),

$$< T > = \frac{1}{3} E_n \tag{26}$$

we get for the ground state

$$\Delta p^2 = \frac{1}{3} \xi_o \frac{\hbar^2}{x_o^2} \tag{27}$$

and therefore

$$\Delta x^2 \ \Delta p^2 = \frac{\hbar^2}{4} \cdot \frac{4}{15} \left(\frac{8}{3} \xi_o^3 + 1 \right) = 1.0186 \cdot \frac{\hbar^2}{4} \tag{28}$$

$$\Delta x \ \Delta p = 1.0093 \ \frac{\hbar}{2}$$

which shows that for the exact ground state wave function, although it is not a minimun wave packet, the uncertainty product $\Delta x \Delta p$ exceeds the minimum value only slightly.

If we were to make a variational approximation for the ground state, taking as a trial function the normalized Gaussian

$$\psi (x) = \left(\frac{\alpha}{\Pi} \right)^{1/4} e^{- \alpha x^2/2} \tag{29}$$

we get for the mean value of the energy

$$< H > = < T > + < V >$$

$$F \ x_o \ \{ \alpha \ x_o^2 + (\Pi \alpha)^{-1/2} \ x_o^{-1} \} \tag{30}$$

The best choise for α is then

$$\alpha = \pi^{-1/3} x_o^{-2} \tag{31}$$

and the approximate ground state energy

$$E'_o = \frac{3}{(4\pi)}^{1/3} F x_o = 1.29 \quad F \quad x_o \tag{32}$$

The mean value of the potential energy satisfies, according to (30)

$$< V > = \frac{2}{3} E'_o$$

that is, we have the same ratio of potential to kinetic energy as in the exact ground state.

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