Journal of Nuclear Energy, Vol. 23, pp. 643 to 654. Pergamon Press 1969. Printed in Northern Ireland

OPERATOR FORMULATION OF RESONANCE ABSORPTION OF NEUTRONS*

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(*Received* 28 *April* 1969)

Abstract—The use of operational calculus allows to obtain exact expressions for the resonance integral and the intermediate resonance parameters to be obtained. These expressions are obtained for the more general case of an heterogeneous system of one resonance absorber with N admixed moderators and M moderators. The trial functions for the fluxes needed in the calculus are obtained by means of a linear approximation for the slowing down operators. There, the intermediate resonance parameters are introduced with physical meaning. Three approximations are used for calculating the resonance integral, and they are applied to several resonances in homogeneous and heterogeneous systems. Then the three approximations are compared with Monte Carlo calculations. One of them is found to be in very good agreement with Monte Carlo results. Finally, the general formulation given here includes many of the I-R methods developed up to the present.

1. INTRODUCTION

It has been shown (B. R. Sehgal (1965) and B. R. Sehgal and R. Goldstein (1966)) that the standard N-R or W-R approximations for calculating the resonance integral are many times not useful. For those cases it is quite convenient to introduce intermediate resonance (I-R) methods.

Here it is developed an operational method of the kind mentioned.

The problem is solved for the general case of an heterogeneous two-region cell of one resonance absorber with N admixed moderators and M moderators.

Exact formulas for the resonance integral and the intermediate parameters are obtained under the following assumptions: (i) The resonances are enough separated as to be considered isolated. (ii) Spatially flat flux in each region: lump and moderator. (iii) The validity of the rational approximation (including Dancoff's factor) for the lump-escape probability. (iv) Temperature and interference scattering are not considered. (Although it is easy to include them.)

In paragraph 3 a linear approximation for the slowing down operators allows the trial functions $\psi_f^{(1)}$ and $\psi_m^{(1)}$ to be easily obtained.

Then three approximate methods are developed for calculating the resonance integral and the intermediate resonance parameters.

The three methods mentioned above are applied to the 6·68 eV resonance of ^{238}U in heterogeneous systems with ^{238}U-Metal and ^{238}UO$_2$ square lattices and to ten resonances of ^{238}U in homogeneous systems. The results are compared with Monte Carlo calculations and the agreement of one of the three approximations is quite good.

In paragraph 6 we analyse in detail the numerical results obtained and in paragraph 7 we give a summary and conclusions of the formulation and the approximations made.

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* This work was partially sponsored by the Consejo Nacional de Investigaciones Científicas y Técnicas of Argentina.

2. MULTINUCLIDE HETEROGENEOUS SYSTEMS

Let us consider a cell composed by a fuel lump surrounded by an external moderator.

σ_a, σ_p, σ_r are the microscopic cross sections for absorption, potential scattering, and resonance scattering of the fuel.

The fuel is admixed with N moderators of microscopic cross sections σ_i ($i = 1, \ldots, N$) per absorber atom of the fuel. Interference scattering is neglected, so:

$$\sigma_s = \sigma_r + \sigma_p \qquad \text{and} \qquad \sigma_T = \sigma_a + \sigma_s + \sum_{i=1}^{N} \sigma_i. \tag{1}$$

The external region has M moderators whose cross sections s_j, ($j = 1, M$) are equal to $\sum_j V_m/N_f V_f$, where $\sum_j$ is the macroscopic cross section of the jth moderator, V_m/V_f is the volume ratio, and N_f is the number density of the resonance absorber. The fuel and moderator-escape probabilities in the lattice cell are $P_f{}^*$ and $P_m{}^*$ respectively.

Assuming zero temperature, and the flat spatial behavior of the flux the neutron-balance equations for a two region problem may be written (J. CHERNICK *et al.* (1955)):

$$\sigma_T(u)\psi_f(u) = (1 - P_f{}^*) \cdot [K_f(\sigma_s\psi_f) + \sum_i \sigma_i K_{am_i}(\psi_f)] + P_m{}^* \sum_j s_j K_{m_j}(\psi_m) \tag{2}$$

and

$$s_m\psi_m(u) = P_f{}^*[K_f(\sigma_s\psi_f) + \sum_i \sigma_i K_{am_i}(\psi_f)] + (1 - P_m{}^*)\sum_j s_j K_{m_j}(\psi_m). \tag{3}$$

Where $\psi_f(u)$ and $\psi_m(u)$ are the mean flux per unit lethargy in each region and $s_m = \sum_{j=1}^{M} s_j$. The slowing down K operators are

$$K_l(\varphi) = \int_{u-\Delta_l}^{u} \frac{e^{-(u-u')}}{1-\alpha_l}\,\varphi(u')\,du' \tag{4}$$

and

$$\alpha_l = [(A_l - 1)/(A_l + 1)]^2; \qquad \Delta_l = \ln \alpha_l^{-1} \tag{5}$$

with A_l the atomic mass of the lth nuclide in neutron mass units. We use for the fuel escape-probability $P_f{}^*$ the rational approximation corrected by Bell's factor given by LESLIE (1965) and for the corresponding to the moderator $P_m{}^*$, the value obtained through the reciprocity relation:

$$P_f{}^*\sigma_T = P_m{}^* s_m. \tag{6}$$

Under such conditions equations (2) and (3) become

$$(s + \sigma_T)\psi_f(u) = K_f[\sigma_s(u)\psi_f(u)] + \sum_{i=1}^{N} \sigma_i K_{am_i}[\psi_f(u)] + \frac{s}{s_m}\sum_{j=1}^{M} s_j K_{mj}[\psi_m(u)] \tag{7}$$

$$(s + \sigma_T)\psi_m(u) = \frac{s}{s_m}\left\{K_f[\sigma_s(u)\psi_f(u)] + \sum_{i=1}^{N} \sigma_i K_{am_i}[\psi_f(u)]\right\} + \frac{s + \sigma_T(1 - s/s_m)}{s_m}\sum_{j=1}^{M} s_j K_{m_j}[\psi_m(u)] \tag{8}$$

where $s = a(1 - C)/N_f l_f$; a, C and l_f are the Bell factor, the Dancoff factor; and the mean chord length. For Dancoff's factor it is used SAUER'S approximation (1963).

Equations (7) and (8) may be written in an operator form:

$$H_f(\psi_f) = (s/s_m)\sum_{j=1}^{M} s_j K_{m_j}(\psi_m) \tag{9}$$

$$H_m(\psi_m) = K_f(\sigma_s\psi_f) + \sum_{i=1}^{N} \sigma_i K_{am_i}(\psi_f) \tag{10}$$

and consequently, the operators H_f and H_m are defined as:

$$H_f = \sigma - K_f\sigma_s - \sum_{i=1}^{N} \sigma_i K_{am_i} \tag{11}$$

$$H_m = (s_m\sigma/s) - s^{-1}(\sigma - \sigma_T s/s_m)\sum_{j=1}^{M} s_j K_{m_j} \tag{12}$$

with $\sigma = \sigma_T + s$.

If we write H_f as

$$H_f = H_f^{(0)} + H_f^{(1)} \tag{13}$$

it is possible to write equation (9) as follows:

$$\psi_f(u) = H_f^{(0)^{-1}}\left[(s/s_m)\sum_{j=1}^{M} s_j K_{m_j}(\psi_m)\right] - H_f^{(0)^{-1}} H_f^{(1)}(\psi_f). \tag{14}$$

We look for a decomposition of H_f such that the term $H_f^{(1)}$ might be neglected in a first order approximation; in this way some function $\psi_f^{(1)}$, which will be called 'trial function', is related to $H_f^{(0)}$ by

$$H_f^{(0)}\psi_f^{(1)} = (s/s_m)\sum_{j=1}^{M} s_j K_{m_j}(\psi_m). \tag{15}$$

This equation does not define completely $H_f^{(0)}$, so we impose a multiplicative condition on $H_f^{(0)}$:

$$H_f^{(0)} = \psi_f^{(1)^{-1}}(s/s_m)\sum_{j=1}^{M} s_j K_{m_j}(\psi_m) \tag{16}$$

but previously $\psi_f^{(1)}$ must be determined. The same holds for H_m and the corresponding $\psi_m^{(1)}$.

3. EXPRESSIONS FOR THE TRIAL FUNCTIONS $\psi_f^{(1)}$ AND $\psi_m^{(1)}$

We will make a survey of the different approximations used in connection with resonance absorption.

(a) *N-R approximation*

The resonance is so narrow compared to the mean energy loss per collision that its contribution to the slowing down integral is negligible. Outside the resonance it is $\psi(u) \to \psi_{asimpt}$; $\sigma_s(u) \to \sigma_p$ and the collision density $F(u) = K[\sigma_s\psi]$ may be approximated by:

$$F_{NR}(u) \cong \sigma_p\psi_{asimpt}.$$

(b) *W-R approximation*

The integration range of the K operator is so narrow compared to the resonance width that in equation (4) we can put:

$$\varphi(u') \cong \varphi(u) \qquad \text{and} \qquad e^{-(u-u')} \cong 1$$

and consequently

$$F_{WR}(u) = \sigma_s(u)\psi(u).$$

(*c*) *I-R approximation*

Since the two above approximations represent the extremes of absorber energy degradation, it is hoped that the actual resonance absorption will lie between the values calculated from them. That is why we put:

$$F_{iR}(u) = \lambda\sigma_p\psi_{asimpt} + (1 - \lambda)\sigma_s(u)\psi(u) \tag{17}$$

where λ may be thought (FORTI (1964) as the average probability for a neutron that had a scattering collision with an absorber atom to be slowed down to energies outside the resonance line. In this way and normalizing $\psi_{asimpt} = 1$ we write the slowing down operators in the form:

$$\begin{aligned} K_f(\sigma_s\psi_f) &= \lambda\sigma_p + (1 - \lambda)\sigma_s\psi_f \\ K_{am_i}(\psi_f) &= \kappa_1 + (1 - \kappa_i)\psi_f \\ K_{m_j}(\psi_m) &= \mu_j + (1 - \mu_j)\psi_m. \end{aligned} \tag{18}$$

These linear approximations for the K operators are very useful, since they allow us to obtain simple expressions for the trial funclions $\psi_f^{(1)}$ and $\psi_m^{(1)}$. By using expressions (18) into the flux balance equations (7) and (8) we obtain the following trial functions for the fluxes

$$\psi_f^{(1)} = \frac{1 + x^2}{\gamma^2 + x^2} \tag{19}$$

$$\psi_m^{(1)} = \frac{\varepsilon^2 + x^2}{\gamma^2 + x^2} \tag{20}$$

where

$$\gamma^2 = 1 + \frac{\sigma_0}{\lambda\sigma_p + \sum\limits_{i=1}^{N} \kappa_1\sigma_1 + s\mu'} \cdot \left(\frac{\Gamma_\gamma + \lambda\Gamma_n}{\Gamma}\right) \tag{21}$$

$$\mu' = \frac{\sum\limits_{j=1}^{M} \mu_j s_j}{\sum\limits_{j}^{M} \mu_j s_j(1 - s/s_m) + s}$$

$$\varepsilon^2 = 1 + \mu'(\gamma^2 - 1)(1 - s/s_m). \tag{22}$$

Here x equals $2/\Gamma \,.\, (E - E_r)$; σ_0, Γ_n, Γ_γ, Γ are the peak cross section, neutron, capture and total widths, and E_r is the resonance energy.

4. THE EXACT RESONANCE INTEGRAL

With the trial function $\psi_f^{(1)}$ we can get $H_f^{(0)}$ and then ψ_f.

From (16) and (19) we have:

$$H_f^{(0)} = \frac{\sigma_a + \lambda\sigma_p + \sum\limits_{i}^{N} x_i\sigma_i + s\mu'}{\lambda\sigma_p + \sum\limits_{i}^{N} \kappa_i\sigma_i + s\mu'} \cdot \frac{S}{S_m} \sum_{j}^{M} s_j K_{m_j}(\psi_m). \tag{23}$$

From equations (23), (13), (11) and (14) we have:

$$\psi_f = \psi_f^{(1)}\left\{1 + \left(\lambda\sigma_p + \sum_i^N \kappa_i\sigma_i + s\mu'\right)^{-1} \cdot \left[\lambda(\sigma_s\psi_f - \sigma_p) + K_f(\sigma_s\psi_f) - \sigma_s\psi_f + \sum_i^N \kappa_i\sigma_i(\psi_f - 1) + \sum_i^N \sigma_i K_{am_i}(\psi_f) - \sum_i^N \kappa_i\psi_f + s\mu'(\psi_f - 1) + (s/s_m) \times \sum_j^M s_j\, K_{m_j}(\psi_m) - s\psi_f\right]\right\}. \tag{24}$$

In a first order approximation the resonance integral is given by

$$I^{(1)} = [\sigma_a, \psi_f^{(1)}] = I_0/\gamma \tag{25}$$

where $[f, g]$ denotes scalar product, and I_0 is the infinite-dilution resonance integral:

$$I_0 = \pi\sigma_0\Gamma_\gamma/2E_r.$$

The exact resonance integral is given by

$$I = [\sigma_a, \psi_f]. \tag{26}$$

Equating $I^{(1)} = I$ we obtain the exact expressions for the resonance parameters λ, κ_i, μ':

$$\left.\begin{aligned} \lambda &= 1 - \frac{\displaystyle\int_{-\infty}^{\infty} \sigma_a(x)\psi_f^{(1)}(x)\{\sigma_p - K_f[\sigma_s\psi_f]\}\,\mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \sigma_a(x)\psi_f^{(1)}(x)[\sigma_p - \sigma_s(x)\psi_f]\,\mathrm{d}x} \\ \kappa_i &= 1 - \frac{\displaystyle\int_{-\infty}^{\infty} \sigma_a(x)\psi_f^{(1)}(x)\{1 - K_{am_i}[\psi_f(x)]\}\,\mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \sigma_a(x)\psi_f^{(1)}(x)[1 - \psi_f(x)]\,\mathrm{d}x} \\ \mu' &= 1 - \frac{\displaystyle\int_{-\infty}^{\infty} \sigma_a(x)\psi_f^{(1)}(x)\left\{1 - s_m^{-1}\sum_{j=1}^{M} s_j K_{m_j}[\psi_m(x)]\right\}\mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \sigma_a(x)\psi_f^{(1)}(x)[1 - \psi_f(x)]\,\mathrm{d}x} \end{aligned}\right\} \tag{27}$$

and from them the exact resonance integral:

$$I = I_0/\gamma. \tag{28}$$

It is interesting to note that for homogeneous systems the above equations for λ, κ_i and μ' reduce to those obtained by ISHIGURO (1968) who used an algebraic procedure.

Since the resonance parameters λ, κ and μ' were obtained by equating $I^{(1)}$ with the exact resonance integral I, then the equation (27) for λ, κ and μ' are exact. (This means that the exact numerical values for them can be obtained by an iterative procedure.)

Hence the first order result given by equation (25) gives the exact value of the effective resonance integral I.

5. THREE APPROXIMATIONS FOR THE RESONANCE INTEGRAL

If the exact $\psi_f(x)$ and $\psi_m(x)$ were known, by replacing them in equations (27) we should obtain the exact λ, κ_i and μ'. As this is not the case we will do three different approximations which will give us quite good results:

Method (*a*)

(i) In equations (27) we replace $\psi_f(x)$ and $\psi_m(x)$ by the trials functions $\psi_f^{(1)}(x)$ and $\psi_m^{(1)}(x)$.

(ii) The integral K operators are replaced by (GOLDSTEIN (1962))

$$K(\varphi) = (1/\delta) \int_x^{x+\delta} \varphi(x')\, dx' \tag{29}$$

with $\delta = (2/\Gamma)E_r(1 - \alpha)$. Then we get for λ, κ_i, μ' the following approximate expressions:

$$\left.\begin{aligned} \lambda &= 1 - X^f \\ \kappa_i &= 1 - X_i^{am};\ i = 1, \dots N \\ \mu' &= \frac{\sum\limits_j^M s_j(1 - X_j^m)}{\sum\limits_j^M [s_j - (1 - s/s_m)s_j X_j^m]} \end{aligned}\right\} \tag{30}$$

with

$$X^q = \frac{\operatorname{arctg} \mathrm{x}^q}{\mathrm{x}^q}\,; \qquad \mathrm{x}^q = \frac{\delta^q}{2\gamma} \tag{31}$$

where the q index stand for f, am_i, m_j which corresponds to the fuel, the admixed moderators, and the outside moderators.

Method (*b*)

We keep hypotheses (i) and (ii) of Method (a), but we choose:

$$H_f^{(0)} = \sigma \Rightarrow H_f^{(1)} = K_f \sigma_s + \sum_{i=1}^{N} \sigma_i K_{am_i}. \tag{32}$$

With this choice of $H_f^{(0)}$ and equating $I^{(1)}$ and a second order approximation $I^{(2)}$ to the resonance integral, we obtain for the intermediate resonance parameters the same expressions as equations (30) but now

$$\mathrm{x}^q = \frac{\delta^q}{\gamma_1 + \gamma} \tag{33}$$

where $\gamma_1 = \gamma(\lambda = \kappa_1 = \dots \kappa_i = \dots \mu_j = 1)$, for $i = 1, \dots, N$ and $j = 1, \dots, M$.

It must be pointed out that these results are the same as those obtained by SEHGAL and GOLDSTEIN (1966) though their formulae are valid for the particular case of $M = N = 1$, i.e. one moderator and one admixed moderator.

Method (*c*)

(i) We use again equations (27) replacing ψ_f and ψ_m by the trial functions $\psi_f^{(1)}$ and $\psi_m^{(1)}$

(ii) We use the actual (exact) expression for the K operators that is:

$$K(\varphi) = \frac{\dfrac{\Gamma}{2E_r}\mathrm{x} + 1}{\delta} \int_{\mathrm{x}}^{\mathrm{x}+\delta/\alpha} \frac{\varphi(\mathrm{x}')\mathrm{d}x'}{\left(\dfrac{\Gamma}{2E_r}\mathrm{x}' + 1\right)^2}. \tag{34}$$

Then the resonance parameters λ, κ_i and μ' become:

$$\lambda = 1 - [C_1 - C_2F_2(\alpha_\lambda, \delta_\lambda, \gamma) + C_3F_3(\alpha_\lambda, \delta_\lambda, \gamma)]$$

$$\kappa_i = 1 - [C_1 - C_2F_2(\alpha_{am_i}, \delta_{am_i}, \gamma) + C_3F_3(\alpha_{am_i}, \delta_{am_i}, \gamma)]$$

$$\mu' = 1 - \frac{\gamma^2 - \varepsilon^2}{\gamma^2 - 1}\left[C_1 - C_2\sum_j^M \frac{s_j}{s_m} F_2(\alpha_{m_j}, \delta_{m_j}, \gamma) + C_3\sum_j^M \frac{s_j}{s_m} F_3(\alpha_{m_j}, \delta_{m_j}, \gamma)\right] \tag{35}$$

where

$$C_1 = \frac{2A^2}{1 + A^2}; \qquad C_2 = 2A\frac{3 - A^2}{(1 + A^2)^2}; \qquad C_3 = 2\frac{1 - 3A^2}{(1 + A^2)^2}$$

$$A = b\gamma; \qquad b = \Gamma/2E_r$$

$$F_2(\alpha, \delta, \gamma) = \frac{\gamma}{2\delta}\ \ln\frac{(1 + \alpha)^2 + \delta^2/\gamma^2}{4}$$

$$F_3(\alpha, \delta, \gamma) = \frac{\gamma}{\delta}\cdot\mathrm{Arctg}\frac{\delta}{\gamma(1 + \alpha)}. \tag{36}$$

It is easy to see that if we do $b = 0$ and $\alpha = 1$, we obtain for equations (35) the same expressions as those given by equations (30)

This must not surprise us since the K operator of equation (29) comes from doing $b = 0$ and $\alpha = 1$ in equation (34)

6. COMPARISON OF THE THREE APPROXIMATIONS WITH MONTE CARLO CALCULATIONS

The three approximation methods were applied to the 6·68 eV resonance of ^{238}U for different moderator masses, rod radii and V_m/V_f relations.

Table 1 contains resonance integral calculations for ^{238}U-Metal fuel lattices and Table 2 for ^{238}UO$_2$ also.

In these tables there are also values obtained by the Monte Carlo code 'Repetitious' as given by SEHGAL and GOLDSTEIN (1966).

The number densities and the potential scattering cross sections were taken from this last reference too. The resonance parameters (E_r, Γ_n, Γ_γ) used in the 3 methods and in MC calculations, are those given by DRESNER (1960).

The corresponding values of λ, κ_i and μ' are shown in Tables 3 and 4.

In Table 3 there is no value for κ since we deal with ^{238}U-Metal lattices and there is no admixed moderator. The values of S, the 'effective' scattering cross section of the fuel lump, are also shown in Tables 3 and 4.

Table 1.—Comparison of resonance integrals
Heterogeneous systems: ^{238}U-metal square lattices, 6·68 eV resonance. Interference scattering absent; temperature: 0°K

No.	MOD mass	Vm/Vf	Rod radius (cm)	Method (a)	Method (b)	Method (c)	Monte Carlo
1	1	0·5	0·762	2·17	2·25	2·30	2·30 ± 0·01
2		0·5	0·3175	2·41	2·52	2·58	2·64 ± 0·02
3		4·0	0·762	3·51	3·52	3·54	3·54 ± 0·05
4		4·0	0·3175	5·0	5·02	5·04	5·01 ± 0·07
5	9	4·0	0·762	3·05	3·13	3·15	3·14 ± 0·02
6		4·0	0·3175	3·97	4·07	4·11	4·15 ± 0·02
7		16·0	0·762	3·53	3·55	3·56	3·63 ± 0·05
8		16·0	0·3175	5·25	5·27	5·29	5·33 ± 0·06
9	12	4·0	0·762	1·91	2·27	2·25	2·35 ± 0·01
10		4·0	0·3175	2·14	2·58	2·56	2·75 ± 0·01
11		40·0	0·762	3·49	3·51	3·52	3·57 ± 0·05
12		40·0	0·3175	5·17	5·19	5·21	5·29 ± 0·06
13	56	20·0	0·3175	3·65	4·20	4·08	4·49 ± 0·03
14		120·0	0·3175	5·39	5·34	5·35	5·34 ± 0·08

Table 2.—Comparison of resonance integrals
Heterogeneous systems: ^{238}UO$_2$. Square lattices, 6·68 eV resonance. Interference scattering absent; temperature: 0°K

No.	MOD mass	Vm/Vf	Rod. radius (cm)	Method (a)	Method (b)	Method (c)	Monte Carlo
15	1	0·5	0·3175	4·08	4·18	4·24	4·30 ± 0·03
16		4·0		7·55	7·56	7·58	7·57 ± 0·12
17	9	2·0		5·03	5·18	5·22	5·37 ± 0·03
18		16·0		7·92	7·93	7·96	7·95 ± 0·12
19	12	4·0		4·46	4·70	4·71	4·97 ± 0·02
20		40·0		7·83	7·85	7·87	7·88 ± 0·12

Table 3.—Comparison of resonance parameters.
Heterogeneous systems: ^{238}U-metal square lattices, 6·68 eV resonance. Interference scattering absent; temperature: 0°K

	Method (a)			Method (b)			Method (c)		
No.	λ	μ	S	λ	μ	S	λ	μ	S
1	0·002	0·800	7·22	0·003	0·849	7·43	0·006	0·870	7·53
2	0·002	0·793	9·14	0·003	0·830	9·48	0·007	0·852	9·67
3	0·004	0·965	15·93	0·006	0·969	15·94	0·009	0·974	15·94
4	0·009	0·952	32·83	0·010	0·955	32·85	0·013	0·961	32·91
5	0·003	0·809	14·34	0·005	0·842	14·47	0·008	0·851	14·50
6	0·006	0·774	25·30	0·007	0·803	25·72	0·010	0·813	25·87
7	0·004	0·948	16·39	0·006	0·955	16·39	0·009	0·959	16·39
8	0·010	0·931	36·87	0·011	0·936	36·90	0·014	0·942	36·93
9	0·001	0·509	8·89	0·003	0·641	10·02	0·006	0·633	9·95
10	0·002	0·489	11·63	0·004	0·607	13·64	0·007	0·598	13·51
11	0·004	0·936	16·18	0·006	0·946	16·19	0·009	0·949	16·19
12	0·009	0·918	36·23	0·011	0·925	36·26	0·014	0·930	36·29
13	0·005	0·471	35·25	0·009	0·600	36·51	0·010	0·570	36·27
14	0·010	0·906	38·78	0·011	0·917	38·79	0·014	0·920	38·79

TABLE 4.—COMPARISON OF RESONANCE PARAMETERS
Heterogeneous systems: $^{238}UO_2$ square lattices, 6·68 eV resonance. Interference scattering absent; temperature: 0°K

No.	Method (a)				Method (b)				Method (c)			
	λ	κ	μ	S	λ	κ	μ	S	λ	κ	μ	S
15	0·006	0·403	0·868	20·42	0·008	0·455	0·884	20·72	0·011	0·464	0·896	20·45
16	0·020	0·611	0·968	68·65	0·021	0·621	0·969	68·67	0·024	0·640	0·973	68·74
17	0·009	0·477	0·772	36·19	0·012	0·523	0·796	36·93	0·014	0·530	0·805	37·17
18	0·021	0·625	0·957	77·02	0·023	0·635	0·958	77·03	0·026	0·653	0·962	77·07
19	0·007	0·435	0·690	31·11	0·010	0·498	0·731	32·37	0·012	0·498	0·734	32·46
20	0·020	0·622	0·949	75·72	0·023	0·633	0·952	75·74	0·026	0·650	0·955	75·77

Observing the values of Tables 1 and 2 we see that method (a) always underestimates the resonance capture.

For equal moderator mass systems the difference increases when V_m/V_f decreases, and for equal V_m/V_f the difference increases when the moderator mass increases.

The same qualitative conclusions hold for method (b) except that its errors are much smaller.

Method (c), gives for almost all the cases better results than method (b). However, both methods are essentially in good agreement with Monte Carlo calculations in all the cases.

From Tables 3 and 4 we see that the values of λ, κ and μ' as given by method (c) are in most cases greater than those given by method (b). That means that the resonance as seen by method (c) becomes more N-R than it is seen by method (b).

The three approximations were also applied to homogeneous systems (Table 5), that is, mixtures of ^{238}U with moderator of masses 1, 12, 16 and 56 always with scattering cross section $\sigma_{am} = 100b$.

The resonance parameters (E_r, Γ_γ, Γ_n) used for the calculations are those reported by NORDHEIM (1961).

The values obtained for the resonance integrals may be compared against Monte Carlo calculations given by SEHGAL (1965) which appear in the same table.

The comparison with Monte Carlo calculations shows that methods (b) and (c) give quite good results.

For low-energy resonance (6·7 eV. 21·0 eV) method (a) underestimates resonance capture. This agrees with the fact that the approximation $\alpha = 1$ and $b = 0$ in equation (35) used by method (a) fails at low-energy resonances.

Table 6 contains the intermediate resonance parameters corresponding to those resonance integrals given in Table 5. The general remarks made for the resonance parameters of heterogeneous systems are valid for Table 6 too.

Summarising we may say that, in general, method (c) compares more favourably with Monte Carlo results than methods (b) or (a). However, it is necessary to take into account that for wide resonances (such as 6·7 eV, 21 eV), high moderator masses and high absorber concentrations, the values of method (c) (as well as those of methods (a) and (b)) for the resonance integral are sometimes too low. Examples of that kind are calculations nos. 9, 10, 13, 17, 24 and 28. This means that care must be taken in lattice cells with strong absorption and low moderator power.

The numerical values of resonance integrals calculated by method (a) and (b)

TABLE 5.—COMPARISON OF RESONANCE INTEGRALS
Homogeneous systems; absorber ^{238}U interference scattering absent;
temperature: 0°K

No.	Resonance energy	Mod. mass	Method (a)	Method (b)	Method (c)	Monte Carlo
21	6·7	1	8·537	8·551	8·596	8·573 ± 0·083
22		12	7·355	7·517	7·537	7·457 ± 0·032
23		16	6·815	7·098	7·055	6·912 ± 0·025
24		56	0·104	2·648	1·680	3·391 ± 0·002
25	21·0	1	3·771	3·772	3·775	3·795 ± 0·054
26		12	3·753	3·752	3·588	3·653 ± 0·026
27		16	3·491	3·492	3·509	3·567 ± 0·022
28		56	2·455	2·670	2·532	2·713 ± 0·006
29	36·9	1	2·911	2·945	2·908	2·948 ± 0·043
30		12	2·790	2·799	2·793	2·783 ± 0·021
31		16	2·740	2·741	2·745	2·741 ± 0·018
32		56	2·143	2·154	2·173	2·215 ± 0·005
33	66·3	1	0·987	0·994	0·986	0·992 ± 0·024
34		12	0·972	0·977	0·972	0·972 ± 0·013
35		16	0·966	0·970	0·966	0·965 ± 0·011
36		56	0·902	0·902	0·903	0·910 ± 0·005
37	81·3	1	0·246	0·246	0·246	0·252 ± 0·015
38		12	0·245	0·245	0·245	0·250 ± 0·008
39		16	0·245	0·245	0·245	0·248 ± 0·007
40		56	0·242	0·242	0·242	0·244 ± 0·004
41	102·5	1	0·681	0·697	0·680	0·690 ± 0·016
42		12	0·672	0·685	0·671	0·680 ± 0·001
43		16	0·668	0·680	0·667	0·673 ± 0·008
44		56	0·627	0·632	0·627	0·635 ± 0·003
45	116·5	1	0·331	0·331	0·331	0·340 ± 0·015
46		12	0·329	0·330	0·329	0·335 ± 0·008
47		16	0·329	0·329	0·329	0·334 ± 0·007
48		56	0·322	0·322	0·322	0·330 ± 0·003
49	189·6	1	0·291	0·299	0·291	0·297 ± 0·008
50		12	0·288	0·295	0·288	0·293 ± 0·004
51		16	0·287	0·294	0·287	0·291 ± 0·004
52		56	0·274	0·278	0·274	0·283 ± 0·002
53	208·5	1	0·196	0·197	0·196	0·196 ± 0·008
54		12	0·195	0·196	0·195	0·194 ± 0·005
55		16	0·195	0·195	0·195	0·193 ± 0·004
56		56	0·190	0·191	0·190	0·189 ± 0·002
57	237·5	1	0·138	0·139	0·138	0·140 ± 0·007
58		12	0·138	0·139	0·138	0·139 ± 0·004
59		16	0·138	0·138	0·138	0·138 ± 0·004
60		56	0·137	0·137	0·137	0·138 ± 0·002

were obtained with the program 'Resonancias' written by CALANDRIA. A more general code 'CAP . RES', written by GRANT for method (b) and (c) provided the numerical values of method (c). Both programs were written in Fortran IV for the GE-625 computer of YPF (Buenos Aires).

The integrals for calculating λ, κ and μ' through equations (27) for method (c) were carried out in the complex-plane by GRANT (1969).

TABLE 6.—COMPARISON OF RESONANCE PARAMETERS
Homogeneous systems; absorber ^{238}U. Interference scattering absent; temperature: 0°K

No.	Method (a)		Method (b)		Method (c)	
	λ	κ	λ	κ	λ	κ
21	0·025	0·909	0·026	0·912	0·029	0·922
22	0·019	0·674	0·023	0·704	0·024	0·708
23	0·016	0·578	0·023	0·627	0·021	0·620
24	0·000	0·000	0·006	0·087	0·005	0·035
25	0·112	0·960	0·106	0·959	0·117	0·963
26	0·103	0·859	0·101	0·858	0·108	0·867
27	0·099	0·818	0·099	0·819	0·104	0·828
28	0·053	0·397	0·078	0·472	0·060	0·423
29	0·139	0·965	0·111	0·960	0·144	0·968
30	0·130	0·877	0·107	0·862	0·135	0·883
31	0·127	0·842	0·106	0·825	0·132	0·849
32	0·084	0·489	0·087	0·495	0·091	0·506
33	0·427	0·986	0·404	0·985	0·430	0·988
34	0·422	0·952	0·401	0·949	0·425	0·957
35	0·420	0·938	0·400	0·934	0·423	0·939
36	0·396	0·800	0·388	0·796	0·399	0·803
37	0·858	0·997	0·858	0·998	0·858	0·997
38	0·857	0·991	0·858	0·991	0·858	0·991
39	0·857	0·989	0·858	0·989	0·858	0·989
40	0·856	0·963	0·857	0·964	0·856	0·963
41	0·421	0·986	0·380	0·984	0·423	0·986
42	0·415	0·951	0·377	0·945	0·419	0·952
43	0·958	0·997	0·376	0·930	0·416	0·938
44	0·391	0·798	0·364	0·784	0·394	0·801
45	0·757	0·996	0·753	0·995	0·758	0·995
46	0·756	0·984	0·753	0·984	0·756	0·984
47	0·755	0·979	0·753	0·979	0·756	0·979
48	0·751	0·934	0·750	0·934	0·752	0·934
49	0·513	0·989	0·474	0·988	0·515	0·989
50	0·510	0·962	0·472	0·958	0·512	0·963
51	0·508	0·952	0·472	0·946	0·510	0·952
52	0·493	0·845	0·463	0·832	0·495	0·847
53	0·740	0·995	0·730	0·995	0·741	0·995
54	0·739	0·983	0·730	0·982	0·740	0·983
55	0·738	0·978	0·730	0·977	0·739	0·978
56	0·733	0·929	0·727	0·927	0·734	0·929
57	0·847	0·997	0·844	0·997	0·847	0·997
58	0·846	0·990	0·844	0·990	0·846	0·990
59	0·846	0·988	0·844	0·987	0·847	0·988
60	0·844	0·960	0·844	0·960	0·845	0·960

7. SUMMARY AND CONCLUSIONS

Three methods were applied to several resonances in homogeneous and heterogeneous systems with an arbitrary number of moderators and admixed moderators in the fuel.

Methods (b) and (c) are in general in closed good agreement with Monte Carlo calculations, though method (c) seems better specially in heterogeneous systems.

For undermoderated systems (low-energy resonances, high absorber concentrations and heavy moderator masses) both methods slightly underestimates the resonance capture. This suggests that the I-R approximation approaches zero resonance capture much faster than Monte Carlo calculations.

We think that in these cases there must be an evident asymmetry in the actual flux distribution of the kind pointed out by GOLDSTEIN (1964). For taking into account for that asymmetry it should perhaps be necessary to introduce a trial function of the kind proposed by POMRAMING and DYOS (1967).

The case of strong absorption and low moderator power is an important problem which holds for example at the very low-energy resonances of the plutonium isotopes, though in these cases up-scattering must also be taken into account.

As for the theory we should stress the development of an operational procedure which allows to obtain the exact resonance integral equation (28).

This operator formulation has an essential unity from a theoretical point of view and in addition it allows to obtain a formalism for the most general case of one or two-region problem with an arbitrary number of nuclear species in both regions. A further generalization might allow a more realistic space-dependence than only two regions.

There are a rather large number of works on the I-R method, but many of them are implicit in the general theory developed here. The following published formulations are included in the theory, each of them valid for more restricted systems.

(1) Equations (27) applied to homogeneous systems give those obtained by ISHIGURO (1968).

(2) Method (b) applied to heterogeneous systems gives the equations of SEHGAL and GOLDSTEIN (1966) which are valid only for the case $M = N = 1$.

(3) The same method applied to homogeneous systems of one absorber with N moderators, reduces to the equations obtained by GOLDSTEIN (1967 b).

(4) Method (a) applied to homogeneous systems, one absorber and one moderator, gives the equations obtained by GOLDSTEIN (1967 a).

Acknowledgments—The authors wish to thank Dr. Elba Pezzoni for her valuable suggestions.

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