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COLLECTIVE TREATMENT OF THE PAIRING HAMILTONIAN

(II). Charge-independent force — Hamiltonian and symmetries

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Abstract: We develop a collective treatment of the problem of neutrons and protons interacting throughout a charge-independent pairing force. The Hamiltonian is expressed in terms of six variables, four of them being angular variables defining orientations in isospace and gauge space, while the remaining two variables represent intrinsic deformation. The symmetries of the problem are discussed in terms of the solutions corresponding to the vibrational limit.

1. Introduction

In a preceding paper [1]) a collective treatment of the pairing interaction between identical particles was developed. The purpose of this second paper is to extend that treatment to the case in which both protons and neutrons are present and interacting through a $J = 0$ charge-independent force acting among them.

It has been suggested [2-5]) that this kind of residual interaction plays an important role in the description of nuclear states in the region $40 \leqq A \leqq 70$.

Far away from closed shells, the system under consideration acquires a permanent distortion in gauge space and in isospace. The first one is associated with the non-conservation of the number of particles while the second one is related to the existence of particles which are a mixture of neutrons and protons [6, 7]). Under these circumstances it is possible to obtain a separation between the intrinsic and collective variables. The intrinsic wave function is obtained by expending [6]) the BCS method used in dealing with a system of identical fermions. The adequacy of this separation was checked against exact results in a two-level model [8]).

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For closed-shell systems, the neutron and proton BCS gaps are zero. The elementary modes of excitation of the system are the pair-addition and pair-subtraction modes, corresponding to fluctuations of the gap parameters around their (zero) equilibrium value.

Most of the available data suggest that an intermediate situation is the one prevailing in nuclei of the f-p shells [3-5]). Therefore, in this paper, we have attempted to develop a macroscopic model including both the rotational and vibrational degrees of freedom associated with an isoscalar pairing force, in analogy with the collective treatment of the quadrupole deformation [9]).

In the identical-particle case, only two variables are needed to specify completely the problem, namely, the gauge angle, which is the variable canonical conjugate to the number-of-particles operator, and an intrinsic deformation that is related to the gap parameter.

In the case in which the system under consideration is composed of protons and neutrons, one extra deformation parameter must be added to the set of intrinsic variables. In addition to the gauge angle, three additional Euler angles must be included in order to define the orientation of the intrinsic system in isospace.

In sect. 2 the collective variables are defined. In sect. 3 the kinetic energy is derived, the transformation from the laboratory to the intrinsic system is discussed, and the quantization of the kinetic energy is worked out. In sect. 4 the general structure of the wave function is given and the symmetries of the problem are discussed. In sect. 5 the invariants that can be constructed in terms of the collective variables are obtained.

2. The collective variables

An important part of the force between the particles can be represented by a $J = 0$, charge-dependent pairing Hamiltonian:

$$H_p = -G \sum_{\mu} P_{\mu} P_{\mu}^{+}, \tag{1}$$

where the operator P_μ creates a pair of nucleons with total angular momentum zero, isospin one and isospin projection μ. The distorted field approximation replaces the force (1) by

$$V_p = -G \sum_{\mu} (d_{\mu}^{*} P_{\mu} + d_{\mu} P_{\mu}^{+}). \tag{2}$$

The deformation of the potential is defined by the three complex parameters d_μ. These parameters are the dynamical variables associated with the charge-independent pairing force.

In this paper we study the symmetries of the pairing correlations arising from the transformation properties of the collective variables. The parameters d_μ will transform† according to

$$d_\mu = e^{2i\phi} \sum_\nu \mathscr{D}^1_{\mu\nu}(\theta_i)\Delta_\nu \tag{3}$$

where the quantities Δ_ν are referred to the intrinsic system. In (3) ϕ is the gauge angle [refs. [1,2])] and θ_i are the three Euler angles in isospace††.

The fact that there is a total of six collective variables and that the transformation (3) depends on four angular variables, suggests that there exists an intrinsic system in which the deformation is characterized by only two parameters.

Let a_j and b_j denote the real and imaginary part of the cartesian components of the vector $\boldsymbol{d}$:

$$d_j = a_j + ib_j \begin{cases} d_0 = a_z + ib_z \\ d_1 = -\dfrac{1}{\sqrt{2}}\{(a_x - b_y) + i(b_x + a_y)\} \\ d_{-1} = \dfrac{1}{\sqrt{2}}\{(a_x + b_y) + i(b_x - a_y)\}. \end{cases} \tag{4}$$

More precisely, let $d_j^{(L)}$, $a_j^{(L)}$, $b_j^{(L)}$ denote the collective variables as referred to the lab system K_L. Since rotations in charge and gauge spaced commute with each other, the transformation to an intrinsic system can always be performed in two steps. The first one consists in a rotation in isospace leading to an intermediate frame K_ω, in which the components of the deformation are denoted by $d_j^{(\omega)}$, $a_j^{(\omega)}$ and $b_j^{(\omega)}$. Secondly, a gauge transformation to the intrinsic system K_I is carried out. In this frame the variables are denoted by Δ_j, α_j and β_j.

For later use it is convenient to discuss some invariants under gauge transformations. Under a rotation in gauge space, the variables α and β transform accordingly to the following equation:

$$\begin{aligned} a_j^{(\omega)} &= \cos 2\phi\alpha_j - \sin 2\phi\beta_j, \\ b_j^{(\omega)} &= \sin 2\phi\alpha_j + \cos 2\phi\beta_j. \end{aligned} \tag{5}$$

By making the appropriate replacements it is easy to check that

$$a_j^{(\omega)}a_k^{(\omega)} + b_j^{(\omega)}b_k^{(\omega)} = \alpha_j\alpha_k + \beta_j\beta_k, \tag{6a}$$

$$\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{\alpha}\times\boldsymbol{\beta}. \tag{6b}$$

The scalar product instead changes to

$$\boldsymbol{a}\cdot\boldsymbol{b} = \tfrac{1}{2}\sin 4\phi(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} - \boldsymbol{\beta}\cdot\boldsymbol{\beta}) + \cos 4\phi(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}). \tag{7}$$

Therefore the gauge transformation can be regarded as one in which the moduli and relative orientation of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are changed in such a way that the vector

† Greek subindices denote spherical components while latin ones refer to cartesian components.
†† The Wigner matrices $\mathscr{D}_{\mu\nu}{}^1(\delta_i)$ are defined according to the conventions of ref. [10]).

product remains constant. The fact that these vectors are changed into an altogether different pair is emphasized by denoting the latter by the corresponding Greek letters.

The system of coordinates $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ have been made orthogonal through a gauge transformation, is the natural system of coordinates to study the present problem. In sect. 3 it is shown that indeed, this system constitutes the set of principal axis.

3. The kinetic energy terms

In this section we restrict the discussion to the case in which the inertial parameters are independent † of the variables d_μ. We thus treat a simpler problem involving similar symmetry considerations as the more general one.

The most general kinetic energy which is real, quadratic in the velocities, invariant under rotations in isospin and gauge spaces and independent of the dynamical variables is of the form

$$T = \tfrac{1}{2}B\sum_\mu |\dot{d}_\mu^{(\mathrm{L})}|^2, \tag{8}$$

with B a real parameter.

If we perform a rotation in isospace to the frame K_ω, the components of the velocities can be written [9])

$$\begin{aligned}\dot{d}_\mu^{(\mathrm{L})} &= \sum_\nu \dot{\mathscr{D}}^1_{\mu\nu} d_\nu^{(\omega)} + \sum_\nu \mathscr{D}^1_{\mu\nu}\dot{d}_\nu^{(\omega)} \\ &= i\sum_{\nu\rho j} q_j \mathscr{D}^1_{\mu\rho}[M_j^{(\omega)}]_{\rho\nu} d_\nu^{(\omega)} + \sum_\nu \mathscr{D}^1_{\mu\nu}\dot{d}_\nu^{(\omega)},\end{aligned} \tag{9}$$

where q_j is the jth component of the angular velocity and $[M_j^{(\omega)}]_{\rho\nu}$, is the ρ, ν matrix element in the three-dimensional representation of the jth component of the angular momentum in the K_ω system. The $M^{(\omega)}$ matrices obey the commutation relations

$$[M_j^{(\omega)}, M_k^{(\omega)}] = -iM_l^{(\omega)}, \qquad j, k, l \qquad \text{cyclic}. \tag{10}$$

If (9) is replaced in (8), the kinetic energy splits into two parts

$$T = T_{\mathrm{rot}} + T'.$$

The first term is equal to

$$T_{\mathrm{rot}} = \tfrac{1}{2}B\sum_{\mu\mu'\rho} \dot{\mathscr{D}}^{1*}_{\mu\mu'}\dot{\mathscr{D}}^1_{\mu\rho} d_{\mu'}^{(\omega)*} d_\rho^{(\omega)} = \tfrac{1}{2}\sum_{lj} q_l q_j \mathscr{I}_{lj}, \tag{11}$$

where

$$\mathscr{I}_{lj} = \tfrac{1}{2}B\sum_{\mu\rho}\{[M_l^{(\omega)}M_k^{(\omega)}]_{\mu\rho} + [M_k^{(\omega)}M_l^{(\omega)}]_{\mu\rho}\} d_\mu^{(\omega)*} d_\rho^{(\omega)} \tag{12}$$

are the moment of inertia of the system.

† This is equivalent to the irrotational approximation of ref. [9]). In this section we follow the corresponding discussion in ref. [9]).

They take a simpler form if we write them in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\mathscr{I}_{kk'} = \begin{cases} B(a_l^{(\omega)2} + a_m^{(\omega)2} + b_l^{(\omega)2} + b_m^{(\omega)2}) & k = k';\ k, l, m \quad \text{cyclic}, \\ -B(a_k^{(\omega)} a_{k'}^{(\omega)} + b_k^{(\omega)} b_{k'}^{(\omega)}) & k \neq k'. \end{cases} \tag{13a, 13b}$$

In order to treat the remaining part of the kinetic energy, we perform a gauge transformation to the frame K_I, which leaves the term (11) unaffected. Thus, the components of the velocities can be written

$$\dot{d}_\mu^{(\omega)} = 2i\dot{\phi}\, e^{2i\phi} \Delta_\mu + e^{2i\phi} \dot{\Delta}_\mu \tag{14}$$

and T' can be divided into five terms

$$T' = T_{\text{vib}} + T_g + T_{\theta_i, g} + T_{\theta_i, \text{vib}} + T_{g, \text{vib}}, \tag{15}$$

such that two of them are diagonal:

$$T_{\text{vib}} = \tfrac{1}{2} B \sum_\mu |\dot{\Delta}_\mu|^2, \tag{16a}$$

$$T_g = \tfrac{1}{2} B 4\dot{\phi}^2 \sum_\mu |\Delta_\mu|^2, \tag{16b}$$

and three represent coupling terms:

$$T_{\theta_i, g} = \tfrac{1}{2} B (2i\dot{\phi}) \sum_{\mu\mu'\rho} (\dot{\mathscr{D}}^{1*}_{\mu\mu'} \mathscr{D}^1_{\mu\rho} - \mathscr{D}^{1*}_{\mu\mu'} \dot{\mathscr{D}}^1_{\mu\rho}) \Delta_\mu \Delta_\rho,$$

$$T_{\theta_i, \text{vib}} = \tfrac{1}{2} B \sum_{\mu\mu'\rho} (\dot{\mathscr{D}}^{1*}_{\mu\mu'} \mathscr{D}^1_{\mu\rho} \Delta^*_{\mu'} \dot{\Delta}_\rho + \mathscr{D}^{1*}_{\mu\mu'} \dot{\mathscr{D}}^1_{\mu\rho} \dot{\Delta}^*_{\mu'} \Delta_\rho),$$

$$T_{g, \text{vib}} = \tfrac{1}{2} B \sum_\mu (2i\dot{\phi})(\dot{\Delta}^*_\mu \Delta_\mu - \dot{\Delta}_\mu \Delta^*_\mu).$$

One can write the coupling terms using the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$T_{\theta_i, g} = 4B\dot{\phi}\boldsymbol{q} \cdot (\boldsymbol{\beta} \times \boldsymbol{\alpha}), \tag{17a}$$

$$T_{\theta_i, \text{vib}} = B\boldsymbol{q} \cdot (\boldsymbol{\alpha} \times \dot{\boldsymbol{\alpha}} + \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}), \tag{17b}$$

$$T_{g, \text{vib}} = -2B\dot{\phi}(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\alpha}} - \boldsymbol{\alpha} \cdot \dot{\boldsymbol{\beta}}). \tag{17c}$$

We now discuss which are the conditions to be imposed on the rotations leading to the system K_ω and K_I in order to bring the kinetic energy as close as possible to its diagonal form.

The transformation to the K_ω system is chosen such as to diagonalize the components of the tensor of inertia $\mathscr{I}_{kk'}$ corresponding to rotations in isospace. By noting that the product $\boldsymbol{a} \times \boldsymbol{b}$ may be brought to lie on the jth axis this being any of the axes of the K_ω frame of reference, we ensure that $\mathscr{I}_{ik} = \mathscr{I}_{jl} = 0$ $(j \neq k \neq 1)$ according to (13b). In other words, the direction of $\boldsymbol{a} \times \boldsymbol{b}$ corresponds to a principal axis of the system. Since two Euler angles are needed to determine the directions of the jth axis, the remaining angle may be used to satisfy the condition $\mathscr{I}_{k,l} = 0$.

By comparing (13a) and (13b) with (6a) we see that the tensor of inertia is invariant under a rotation in gauge space. This fact further justifies the separation of the general transformation into two successive rotations in isospin and gauge space. We thus conclude that if we achieve a diagonalization of the tensor of inertia by means of a rotation in isospace, the diagonalization will not be spoiled by the subsequent gauge transformation.

When equating to zero the expressions (13b), the three non-trivial solutions for the $a_j^{(\omega)}$ and $b_j^{(\omega)}$ are

$$a_x^{(\omega)} = b_x^{(\omega)} = 0, \qquad a_z^{(\omega)}/b_z^{(\omega)} = -b_y^{(\omega)}/a_y^{(\omega)},$$

or

$$d_1^{(\omega)}/d_{-1}^{(\omega)} = 1, \qquad d_1^{(\omega)}/d_0^{(\omega)} = \text{Real}; \tag{18a}$$

$$a_y^{(\omega)} = b_y^{(\omega)} = 0, \qquad a_z^{(\omega)}/b_z^{(\omega)} = -b_x^{(\omega)}/a_x^{(\omega)},$$

or

$$d_1^{(\omega)}/d_{-1}^{(\omega)} = -1, \quad d_1^{(\omega)}/d_0^{(\omega)} = \text{Imag}; \tag{18b}$$

$$a_z^{(\omega)} = b_z^{(\omega)} = 0, \qquad a_x^{(\omega)}/b_x^{(\omega)} = -b_y^{(\omega)}/a_y^{(\omega)},$$

or

$$d_0^{(\omega)} = 0, \qquad d_1^{(\omega)}/d_{-1}^{(\omega)} = \text{Real}. \tag{18c}$$

It follows from eq. (18) that the gauge angle can be separated from all the other collective variables (as assumed in eq. (3)).

In order to make zero the coupling terms (17b) and (17c) we require the gauge transformation to be such that $\beta_l = 0$, according to (13b). Therefore the two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are always directed along those two principal axes of inertia which are contained in the plane perpendicular to the vector $\boldsymbol{\alpha} \times \boldsymbol{\beta}$. Consequently the velocities $\dot{\boldsymbol{\alpha}}$ and $\dot{\boldsymbol{\beta}}$ also lie on the same axis, respectively, and therefore (17b) and (17c) vanish.

The expression (17a), together with (6b), tells us that we shall not be able to uncouple the degrees of freedom associated with rotations in isospin and gauge spaces since $T_{\theta_i, g}$ is a scalar with respect to rotations in both spaces. The existence of an off-diagonal part in the kinetic energy can be geometrically understood on the basis of the fact that both the third rotation in isospace and the gauge transformation take place in the same plane, i.e. the one that contains the two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

If we introduce the new variables Δ and Γ by

$$\begin{aligned} \alpha_l &= \Delta \sin \Gamma, \\ \beta_k &= \Delta \cos \Gamma, \end{aligned} \tag{19}$$

in analogy with the β, γ variables of the quadrupole case [9]), the "classical" kinetic energy is given by

$$\begin{aligned} T &= T_{\text{rot}} + T_{\text{vib}} + T_g + T_{\theta_i, g}, \\ T_{\text{rot}} &= \tfrac{1}{2}B(\Delta^2 q_j^2 + \Delta^2 \sin^2 \Gamma q_k^2 + \Delta^2 \cos^2 \Gamma q_l^2), \end{aligned} \tag{20a}$$

$$T_{\text{vib}} = \tfrac{1}{2}B(\dot{\Delta}^2+\Delta^2\dot{\Gamma}^2), \tag{20b}$$

$$T_g = 2B\Delta^2\dot{\phi}^2, \tag{20c}$$

$$T_{\theta_i, g} = 2B\Delta^2 \sin 2\Gamma\dot{\phi}q_j. \tag{20d}$$

To get the quantum-mechanical counterpart of (20) two equivalent procedures can be used. The first one consists in writing the kinetic energy in terms of the generalized velocities and then use the Pauli prescription [11-13]). Since the $\dot{\Delta}$ and $\dot{\Gamma}$ part of the kinetic energy is uncoupled from the rest, this procedure can be carried out for this part and yields

$$T_\Delta = -\frac{1}{2B}\frac{1}{\Delta^5}\frac{\partial}{\partial\Delta}\Delta^5\frac{\partial}{\partial\Delta}, \tag{21a}$$

$$T_\Gamma = -\frac{1}{2B\Delta^2}\frac{1}{\sin 4\Gamma}\frac{\partial}{\partial\Gamma}\sin 4\Gamma\frac{\partial}{\partial\Gamma}. \tag{21b}$$

This method, however, becomes rather involved for the rotational and gauge parts, since one should use the time derivatives of the Euler angles rather than the angular velocities q_j. We can write instead the kinetic energies in terms of the conjugate moments Q_i, p_ϕ to the angular variables

$$Q_l = \frac{\partial T_{\text{rot}}}{\partial q_l} = B\Delta^2\cos^2\Gamma q_l, \qquad Q_j = \frac{\partial(T_{\text{rot}}+T_{\theta_i, g})}{\partial q_j} = B\Delta^2(q_j+2\sin 2\Gamma\dot{\phi}),$$

$$Q_k = \frac{\partial T_{\text{rot}}}{\partial q_k} = B\Delta^2\sin^2\Gamma q_k, \qquad p_\phi = \frac{\partial(T_{\theta_i, g}+T_g)}{\partial\dot{\phi}} = 2B\Delta^2(2\dot{\phi}+\sin 2\Gamma q_j).$$

After quantization, the quantities Q_j are the operators corresponding to the components of angular momentum along the intrinsic frame in isospace and $p_\phi = -i\partial/\partial\phi$. Solving for q_j, $\dot{\phi}$ and replacing those values of q_l and q_k in eqs. (20a, c, d) we obtain

$$T_{\text{rot}} = \frac{Q_j^2}{2B\Delta^2\cos^2 2\Gamma} + \frac{Q_k^2}{2B\Delta^2\cos^2\Gamma} + \frac{Q_l^2}{2B\Delta^2\sin^2\Gamma}, \tag{21c}$$

$$T_g = -\frac{1}{8B\Delta^2\cos^2 2\Gamma}\frac{\partial^2}{\partial\phi^2}, \tag{21d}$$

$$T_{\theta_i, g} = -\frac{i\sin 2\Gamma}{2B\Delta^2\cos^2 2\Gamma}Q_j\frac{\partial}{\partial\phi}. \tag{21e}$$

4. Wave functions and symmetries

4.1. DEFINITION OF THE WAVE FUNCTIONS

The total collective Hamiltonian will contain a potential energy depending on the variables Δ and Γ, in addition to the kinetic energy terms (21). The eigenfunctions of this Hamiltonian have T, T_z (total isospin and its projection on the laboratory z-axis) and M-number of pairs measured from the closed shell as good quantum numbers; therefore, they may conveniently be expressed as linear combinations:

$$\psi_{TT_z,M} = \sqrt{\frac{2T+1}{8\pi^3}}\, e^{2iM\phi} \sum_K g_K^{T,M}(\Delta, \Gamma)\mathscr{D}^T_{T_z,K}(\theta_i). \tag{22}$$

Two extra quantum numbers associated with Δ and Γ remain as yet unspecified.

This set of eigenfunctions constitutes complete set of states. They are orthonormalized using a volume element

$$d\tau = \tfrac{1}{4}B^3\Delta^5|\sin 4\Gamma| d\phi\, d\Delta\, d\Gamma\, d\Omega. \tag{23}$$

4.2. SYMMETRY PROPERTIES

Although the pairing field specifies completely the $d_\mu^{(L)}$, the variables in (22) are not univocally determined. For definiteness let us choose the jth axis perpendicular to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be the x-axis of the intrinsic system. In this way the largest moment of inertia is associated with this axis and $\alpha_x = \beta_x = 0$. This is a convenient choice for problems involving axial symmetry ($\Gamma = \alpha_z = 0$) since in such cases the z-axis will be aligned with the symmetry axis †.

However, as in the quadrupole case [9, 13]), there are 24 different ways of defining a right-handed intrinsic system in the three-dimensional isospace. Since in addition there are 4 possible values of the gauge angle in the interval in which $\alpha_k = 0$, we have a total of 96 equivalent intrinsic systems.

In analogy with [9]) one may define 4 basic operators which can be used to transform a given intrinsic system into an equivalent one:

$\mathscr{R}_1$ rotation through $\frac{1}{2}\pi$ around the x-axis of the intrinsic system ††,

$\mathscr{R}_2$ rotation through π around the y-axis of the intrinsic system,

$\mathscr{R}_3$ cyclic permutation of the three intrinsic axes,

$\mathscr{G}_0$ rotation through $\frac{1}{2}\pi$ in the gauge space. (24)

Since $\mathscr{R}_1^4 = \mathscr{R}_2^2 = \mathscr{R}_3^3 = \mathscr{G}_0^4 = 1$ the 96 possible transformations between equivalent intrinsic systems are $S(s_1s_2s_3s_4) = \mathscr{R}_1^{s_1}\mathscr{R}_2^{s_2}\mathscr{R}_3^{s_3}\mathscr{G}_0^{s_4}$ with $0 \leqq s_1 \leqq 3$; $0 \leqq s_2 \leqq 1$; $0 \leqq s_3 \leqq 2$; $0 \leqq s_4 \leqq 3$.

As in the quadrupole case there is a redundance in our description implied by the fact that most of the operators (24) accept two possible interpretations †††. On the one

† This corresponds to the convention of ref. [7]). An equivalent possibility is to align the principal axis corresponding to the largest moment of inertia with the y-axis. Both of this conventions are different to the one adopted in ref. [6]) where the largest moment of inertia is associated with the z-axis. These three possibilities correspond to the three possible choices of eq. (18).

†† The two operations $\mathscr{R}_1$ and $\mathscr{R}_2$ are defined in a different way than in ref. [9]), this alternative choice being used so as to be consistent with the conventions taken hitherto.

††† It is important to note that in the present case the transformation $\mathscr{R}_3$ does not accept an implicit form since it brings the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ outside the (l, k) plane, which cannot be achieved by changing the values of Γ and ϕ.

hand, they can be viewed as acting only on the variables ξ_l explicitly implied in the definitions (24); on the other, as changing the values of the remaining variables ξ_i of the system so as to reproduce the same effect. We will refer to the former as the explicit form of the operator and to the latter as the implicit form; we will denote them as $\mathscr{S}_e(\mathscr{R})$ and $\mathscr{S}_i(\mathscr{R})$, respectively. It is important to point out that we only consider the structure of the Hamiltonian and wave function in terms of the collective variables d_μ. The inclusion of additional degrees of freedom, such as quasiparticles, different collective modes, etc., can also be treated, but will in general modify the symmetry of the collective part of the wave function. The present treatment applies to a system in which the intrinsic degrees of freedom are invariant under all symmetry operations, as is expected to be the case for the ground state configuration of doubly even nuclei. For instance, the rotation through $\frac{1}{2}\pi$ around the x-axis of the intrinsic system (operator $\mathscr{R}_1$) may be performed by a change in the Euler angles, yielding the transformation

$$d_y'^{(\omega)} = d_z^{(\omega)}, \qquad d_z'^{(\omega)} = -d_y^{(\omega)},$$

which is also obtained by substituting $\phi' = \phi+\frac{3}{4}\pi$ and $\Gamma' = -\Gamma+\frac{1}{2}\pi$ in eqs. (19) and (5).

TABLE 1
The effect of the different $\mathscr{S}(\mathscr{R})$

Operator	Set $\{\xi_e\}$	$\xi'_e = \mathscr{S}_e(\mathscr{R})\{\xi_e\}$	Set $\{\xi_l\}$	$\xi'_l = \mathscr{S}_l(\mathscr{R})\{\xi_l\}$
$\mathscr{R}_1$	$\theta_1, \theta_2, \theta_3$	three new Euler angles	ϕ, Γ	$\phi' = \phi+\frac{3}{4}\pi$ $\Gamma' = -\Gamma+\frac{1}{2}\pi$ or $\phi' = \phi+\frac{1}{4}\pi$ $\Gamma' = -\Gamma+\frac{3}{2}\pi$
$\mathscr{R}_2$	$\theta_1, \theta_2, \theta_3$	$\theta'_1 = \theta_1$ $\theta'_2 = -\theta_2+\pi$ $\theta'_3 = \theta_3$	ϕ, Γ	$\phi' = \phi+\frac{1}{2}\pi$ $\Gamma' = -\Gamma$ or $\phi' = \phi$ $\Gamma' = -\Gamma+\pi$
$\mathscr{G}_0$	ϕ	$\phi' = \phi+\frac{1}{2}\pi$	$\theta_1, \theta_2, \theta_3$ Γ	$\Gamma' = -\Gamma+\frac{3}{2}\pi$ $\theta'_l = \mathscr{S}_l(\mathscr{R}_1)(\theta_l)$

The second and fourth columns specify the set of variables ξ_e, ξ_l that are associated with each operator. The third and fifth columns give the values of ξ'_e and ξ'_l respectively. The $\mathscr{S}_e(\mathscr{R})\{\theta_l\}$ are defined by

$$\mathscr{D}^I_{M_1M'}(\theta'_i) = \sum_{M_1} \mathscr{D}^I_{MM_1}(\theta_i)\mathscr{D}^I_{M_1M'}(\theta(\mathscr{R}_i)),$$

where $\theta(\mathscr{R})_1 = (\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi)$ and $\theta(\mathscr{R}_2) = (0, \pi, 0)$.

Table 1 summarizes the effect of $\mathscr{S}_e$ and $\mathscr{S}_i$. The constraint

$$\mathscr{S}_e(\mathscr{R}_n)\psi = \mathscr{S}_i(\mathscr{R}_n)\psi \tag{25}$$

ensures that the effect of rotations is independent of which variables we act upon. From a practical point of view, the constraints (25) provide information on the symmetry properties of the function $g_K^{T,M}(\Delta, \Gamma)$. These eliminate some wave functions corresponding to definite combinations of quantum numbers and also reduce the domain of Γ in which the wave functions must be obtained.

From table 1 it is seen that $\mathscr{S}_i(\mathscr{R}_1)$ and $\mathscr{S}_i(\mathscr{R}_2)$ have two possible effects on the variables Γ and ϕ. If the first of the two alternatives is used both for $\mathscr{R}_1$ and $\mathscr{R}_2$ eq. (25) yields, respectively,

$$g_K^{T,M}(\Delta, \Gamma) = (-i)^M \sum_{K'} g_{K'}^{T,M}(\Delta, -\Gamma+\tfrac{1}{2}\pi)\mathscr{D}^T_{K',K}(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, \tfrac{3}{2}\pi) \tag{26}$$

and

$$g_K^{T,M}(\Delta, \Gamma) = (-)^{T+M+K} g_K^{T,M}(\Delta, -\Gamma). \tag{27}$$

Requiring that the two alternatives for $\mathscr{S}_i(\mathscr{R}_1)$ (or $\mathscr{S}_i(\mathscr{R}_2)$) give the same result, one obtains

$$g_K^{T,M}(\Delta, \Gamma) = (-)^M g_K^{T,M}(\Delta, \Gamma+\pi). \tag{28}$$

The same relation can also be obtained by successive application of (27), (26), (27) and again (26). The information provided by $\mathscr{S}_i(\mathscr{G}_0) = \mathscr{S}_e(\mathscr{G}_0)$ is already contained in (26).

Eqs. (26) and (27) imply that the wave function is determined in the domain $0 \leqq \Gamma \leqq 2\pi$ once it is known in $0 \leqq \Gamma \leqq \frac{1}{4}\pi$. All the integrals involving Γ can therefore be restricted to this smaller interval.

It is useful also to consider the constraint $\mathscr{S}_i^2(\mathscr{R}_1) = \mathscr{S}_e^2(\mathscr{R}_1)$. It yields

$$g_K^{T,M}(\Delta, \Gamma) = (-)^{T+M} g_{-K}^{T,M}(\Delta, \Gamma). \tag{29}$$

Thus for $K = 0$, $T+M$ must be even, as pointed out in refs. [5,7]), and no states with $T = 0$ will exist in systems having an odd number of pairs.

Using (29), the wave function (23) can be rewritten as

$$\psi_{TT_zM}(\theta_i; \phi; \Delta, \Gamma) = \sqrt{\frac{2T+1}{16\pi^3}}\, e^{2iM\phi} \sum_{K \geqq 0} g_K^{T,M}(\Delta, \Gamma) \times (1+\delta_{K,0})^{-\frac{1}{2}} (\mathscr{D}^T_{T_z,K}(\theta_i) + (-)^{T+M}\mathscr{D}^T_{T_z,-K}(\theta_i)). \tag{30}$$

5. Invariants

In what follows we discuss the invariants associated with the problem which are necessary for constructing the physical operators. These invariants are obtained utilizing the fact that the eigenfunctions (30) are irreducible tensors transforming as the T_z row of the representation of rank T with respect to rotations in isospace and according to the representation of range M under rotations in gauge space.

5.1. SCALARS

According to (30), an isoscalar carrying a definite value of M is of the form

$$e^{2iM\phi}g_0^{0,M}(\Delta, \Gamma). \tag{31}$$

By application of the symmetry relations (29), (27), (28) and (26) to the $g_0^{0,M}(\Delta, \Gamma)$ we obtain:

$$\begin{aligned} g_0^{0,M}(\Delta, \Gamma) &= (-)^M g_0^{0,M}(\Delta, \Gamma) \\ &= (-)^M g_0^{0,M}(\Delta, -\Gamma) = (-)^M g_0^{0,M}(\Delta, \Gamma+\pi) = (-i)^M g_0^{0,M}(\Delta, -\Gamma+\tfrac{1}{2}\pi), \end{aligned} \tag{32}$$

implying that M has to be an even number and thus that $g_0^{0,M}(\Delta, \Gamma)$ must be an even function of Γ. Therefore it may be expanded in terms of $\cos(n\Gamma)$. Since the third equality in (32) constraints the expansion to even values of n, we can write an expansion of $g_0^{0,M}(\Delta, \Gamma)$ in a power series of $\cos(2\Gamma)$:

$$g_0^{0,M}(\Delta, \Gamma) = \sum_n C_n^M(\Delta)(\cos 2\Gamma)^n. \tag{33}$$

If we consider the lowest terms in (33), the last equality in eq. (32) tells us that $M = 0$ if $n = 0$ and that $M = \pm 2$ if $n = 1$. These terms correspond to the three only isoscalars that may be constructed bilinear in the two variables d_μ and d_μ^*

$$\begin{aligned} g_0^{0,0} &= \Delta^2 = \sqrt{3}\{\boldsymbol{dd}^*\}^{T=0}, \\ e^{4i\phi}g_0^{0,2} &= \Delta^2 h_+ = \Delta^2 e^{4i\phi}\cos 2\Gamma = \sqrt{3}\{\boldsymbol{dd}\}^{T=0}, \\ e^{-4i\phi}g_0^{0,-2} &= \Delta^2 h_- = \Delta^2 h_+^* . \end{aligned} \tag{34}$$

The isoscalars $\Delta^2 h_+$, $\Delta^2 h_-$ are not gauge scalars and, within the model, they correspond to the specific operator for the transfer of an α-particle. The first function, Δ^2, is a scalar in both spaces, as well as the product

$$\Delta^4 h_+ h_- = \Delta^4 \cos^2 2\Gamma. \tag{35}$$

The functions Δ^2 and $\Delta^4 \cos^2 2\Gamma$ play a role analogous to the scalars β^2 and $\beta^3 \cos 3\gamma$ in the case of quadrupole oscillations.

5.2. WAVE FUNCTIONS

The angular part of the wave functions is built utilizing the fact that the basis of the irreducible representations of the rotation group in an n-dimensional space can be built by the tensor product of the coordinate of this space with itself. In the particular case of three dimensions, these functions are the spherical harmonics. This method was used in ref. [14]) in connection with the quadrupole degree of freedom. For a Γ-independent potential, the Schrödinger equation splits into two parts:

$$\begin{aligned} (T' - \Lambda_{\lambda,T,M})\Phi_{T_z}^{\lambda,T,M}(\Gamma, \phi, \theta_i) &= 0, \\ \left(T_\Delta + \frac{1}{\Delta^2}\Lambda_{\lambda,T,M} + V(\Delta) - E_n\right)\mathscr{R}_n^{\lambda,T,M}(\Delta) &= 0, \end{aligned} \tag{36}$$

with

$$\psi_{n,\lambda,T,T_z,M}(\Delta, \Gamma, \phi, \theta_i) = \mathscr{R}_n^{\lambda,T,M}(\Delta)\Phi_{T_z}^{\lambda,T,M}(\Gamma, \phi, \theta_i),$$

where

$$T' = \Delta^2(T - T_\Delta)$$

is independent of Δ and its derivatives. Here the quantum number λ is called the seniority [14-16].

Let us find the solutions of (36). We construct $2T+1$ "basic vectors" $Z_{T_z}^{T,m}(\Gamma, \phi, \theta_i)$, which for a given T are the solutions of the cases corresponding to the lowest values of $|M|$. Since the quantum number m specifies the ϕ-dependence of the Z-functions, $|m|$ must be $\leqq T$. The solution for an arbitrary M may be expressed as a linear combination

$$\Phi_{T_z}^{\lambda,T,M}(\Gamma, \phi, \theta_i) = \sum_m f_{\lambda,T}^{(M-m)}(h_+, h_-)Z_{T_z}^{T,m}(\Gamma, \phi, \theta_i), \tag{37}$$

where the $f_{\lambda,T}^{(M-m)}$ are polynomials in the isoscalars h_+, h_- defined in (34).

The maximum degree of the polynomials is determined by λ. For a given T, there should exist as many basic vectors $Z_{T_z}^{T,m}$ as coupled differential equations (i.e. $2T+1$).

However, only T or $T+1$ are simultaneously involved, since the possible values of m in (37) are restricted by the condition that the parity of M should be the same as the one of m. This may be seen by setting

$$\begin{aligned} f_{\lambda,T}^{(M-m)}(h_+, h_-) &= \sum_{n_+,n_-} C_{n_+,n_-} h_+^{n_+} h_-^{n_-} = \sum_{n_+,n_-} C_{n_+,n_-} e^{4i\phi(n_+-n_-)}(\cos 2\Gamma)^{(n_++n_-)} \\ &= e^{2i(M-m)\phi}(\cos 2\Gamma)^{\frac{1}{2}(M-m)} \sum_{K=0} C_K(\cos 2\Gamma)^{2K}. \end{aligned} \tag{38}$$

In (38) the exponent of cos 2Γ outside the summation must be an integer, therefore implying that M and m should have the same parity. The summation in (37) involves only $T+1$ even values of m if M is even or T odd values of m if M is odd ($|m| \leqq T$). From (38), it also follows that the polynomials $f_{\lambda,T}^{(M-m)}$ have a definite parity in cos 2Γ.

As said before, the functions $Z_{T_z}^{T,m}$ constitute the elementary tensors that play a role in the six-dimensional case, which is analogous to the one played by the spherical harmonics in the three-dimensional case. The $Z_{T_z}^{T,m}$ functions for $T \leqq 2$ are listed in the appendix.

By introducing (37) in (36) we may obtain the differential equations which are obeyed by the polynomials $f_{\lambda,T}^{(M-m)}$, and thereoff recurrence relations for the coefficients C_K can be constructed.

The solutions of (36) are characterized by the number of radial nodes n. Each time that n is increased by one for a given λ, T and M, a pair of particles with $T = 1$ and zero angular momentum is promoted from below to above the Fermi surface in a total configuration with $T = 0$: these states constitute a Δ-band.

5.3. POTENTIAL AND KINETIC ENERGY

A particular case of (36) corresponds to the harmonic potential

$$V(\Delta) = \tfrac{1}{2}C\Delta^2. \tag{39}$$

In this case, the radial wave function $\mathscr{R}_n^{\lambda, T, M}(\Delta)$ are given by well-known associated Laguerre polynomials. Therefore, we have succeeded in constructing at least one complete set of wave functions (30). Aside from the direct applicability of this set to describe the experimental situation, we use it in the construction of the most general Hamiltonian. The potential energy, for instance, has to be both a scalar in isospace and in gauge space and therefore it can be expanded [13, 17]) in terms of the solutions with $M = T = 0$. Since in this case $m = 0$ and $Z_0^{0,0} = 1$, $\Phi_0^{\lambda,0,0}$ is a power series in even powers of $\cos 2\Gamma$. This result could also be directly obtained from the discussion on the variants Δ^2 and $\Delta^4 \cos^2 2\Gamma$ following eq. (35). A general expression for a suitable potential energy can therefore be written as:

$$\begin{aligned} V(\Delta, \Gamma) &= V_0 + V_2\Delta^2 + V_4\Delta^4(1+k_{2,2}h_+h_-) + V_6\Delta^6(1+k_{2,3}h_+h_-) + \dots \\ &= \sum_{r=0}^{\infty} V_{2r}\Delta^{2r}\Big(\sum_{s=0}^{l} k_{s,r}(\cos 2\Gamma)^{2s}\Big), \end{aligned} \tag{40}$$

$$l = \begin{cases} \tfrac{1}{2}r & \text{if } r \text{ is even} \\ \tfrac{1}{2}(r-1) & \text{if } r \text{ is odd,} \end{cases}$$

where V_{2r} and $k_{s,r}$ are arbitrary constants.

To construct a general expression for the kinetic energy we follow the same procedure as in ref. [13]). First we note that each collective coordinate d_μ, or its time derivative, carries an isospin $T = 1$, and $M = 1$ (i.e. transfer quantum number equal to 2). Therefore a quadratic expression in d_μ will carry possible (M, T) values (0, 0); (0, 1); (0, 2); (± 2, 0) and (± 2, 2). Next, we construct total scalars in isospace and gauge space by coupling each (M, T) term with a function $\Phi_{T,M}$ transforming contragradiently. Finally, each of the thirteen possible terms can still be multiplied by an arbitrary function χ of the two invariants Δ^2 and $\Delta^4 \cos^2 2\Gamma$ satisfying the necessary conditions of regularity and analyticity. If we use real χ-functions, then the further condition that the kinetic energy is real reduces the total number of arbitrary functions to eight. The general expression for the kinetic energy is then as follows:

$$\begin{aligned} T = {} & \chi_{00}[\dot{d}\dot{d}^*]^{T=0}Z_0^{0,0} + \chi_{20}\{[\dot{d}\dot{d}]^{T=0}Z_0^{0,0}h_- + [\dot{d}^*\dot{d}^*]^{T=0}Z_0^{0,0}h_+\} \\ & + \chi_{01}[[\dot{d}\dot{d}^*]^{T=1}Z^{1,0}]^{T=0} + \chi_{0,2}^{(1)}[[\dot{d}\dot{d}^*]^{T=2}Z^{2,0}]^{T=0} \\ & + \chi_{0,2}^{(2)}\{[[\dot{d}\dot{d}^*]^{T=2}Z^{2,2}]^{T=0}h_- + [[\dot{d}\dot{d}^*]^{T=2}Z^{2,-2}]^{T=0}h_+\} \\ & + \chi_{22}^{(1)}\{[[\dot{d}\dot{d}]^{T=2}Z^{2,0}]^{T=0}h_- + [[\dot{d}^*\dot{d}^*]^{T=2}Z^{2,0}]^{T=0}h_+\} \\ & + \chi_{2,2}^{(2)}\{[[\dot{d}\dot{d}]^{T=2}Z^{2,2}]^{T=0}h_-^2 + [[\dot{d}^*\dot{d}^*]^{T=2}Z^{2,-2}]^{T=0}h_+^2\} \\ & + \chi_{2,2}^{(3)}\{[[\dot{d}\dot{d}]^{T=2}Z^{2,-2}]^{T=0} + [[\dot{d}^*\dot{d}^*]^{T=2}Z^{2,2}]^{T=0}\}. \end{aligned} \tag{41}$$

It must be noted that also eight real functions of Δ and Γ are involved if the kinetic energy is written as

$$T = \tfrac{1}{2}\sum_K \mathscr{I}_K(\Delta,\Gamma)q_\kappa^2+\tfrac{1}{2}\mathscr{I}_\phi(\Delta,\Gamma)\dot{\phi}^2 + \mathscr{I}_{\phi,l}(\Delta,\Gamma)\dot{\phi}q_l+\tfrac{1}{2}B_\Delta(\Delta,\Gamma)\dot{\Delta}^2+B_{\Delta,\Gamma}(\Delta,\Gamma)\dot{\Delta}\dot{\Gamma}+\tfrac{1}{2}B_\Gamma(\Delta,\Gamma)\dot{\Gamma}^2. \quad (42)$$

6. Conclusions

In this paper we have developed an adiabatic treatment of the degrees of freedom that are associated with the $T = 1$ pairing force. As in the case of the quadrupole force, this treatment assumes the existence of an intrinsic frame. It does not assume, however, that the frequencies associated with the rotations of the intrinsic frame are smaller than the frequencies corresponding to changes in the deformation of the body, and therefore, it should be suitable for application, for example, in the Ni region, where the experimental evidence suggests that we are in a transition region.

We have confined ourselves to the study of the symmetries of the problem and to the derivation of the most general Hamiltonian. This Hamiltonian contains nine functions of Δ^2 and $\Delta^4 \cos^2 2\Gamma$, which have to be specified. In subsequent papers we shall study the solutions corresponding to different assumptions on these functions.

It is plausible that the coupling of the pairing with other degrees of freedom (especially the quadrupole ones) should be simple within the adiabatic hypothesis. Aside from this expectation, we believe that the present study has the formal value of putting the treatment of the isoscalar pairing force within the same collective framework which has been so useful for other degrees of freedom.

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Appendix

$Z_{T_z}^{T,m}$ FUNCTIONS FOR $T \leq 2$

$T = 0$

$$Z_0^{0,0} = 1.$$

$T = 1$

$$Z_{T_z}^{1,1} = i\,e^{2i\phi}\{\cos\Gamma\mathscr{D}^1_{T_z,0}(\theta_i)-\sqrt{\tfrac{1}{2}}\sin\Gamma(\mathscr{D}^1_{T_z,1}(\theta_i)+\mathscr{D}^1_{T_z-1}(\theta_i))\},$$
$$Z_{T_z}^{1,0} = \tfrac{1}{2}\sin 2\Gamma(\mathscr{D}^1_{T_z,-1}(\theta_i)-\mathscr{D}^1_{T_z,1}(\theta_i)).$$

$T = 2$

$$Z^{2,2}_{T_z} = e^{4i\phi}\{\sqrt{\tfrac{2}{3}}(\cos^2\Gamma+\tfrac{1}{2}\sin^2\Gamma)\mathscr{D}^2_{T_z,0}(\theta_i)+\tfrac{1}{2}\sin^2\Gamma(\mathscr{D}^2_{T_z,2}(\theta_i)+\mathscr{D}^2_{T_z-2}(\theta_i))$$
$$-\tfrac{1}{2}\sin 2\Gamma(\mathscr{D}^2_{T_z,1}(\theta_i)+\mathscr{D}^2_{T_z,-1}(\theta_i))\},$$
$$Z^{2,1}_{T_z} = \sqrt{\tfrac{1}{6}}e^{2i\phi}\sin 2\Gamma\{\cos\Gamma(\mathscr{D}^2_{T_z,-1}(\theta_i)-\mathscr{D}^2_{T_z,1}(\theta_i))+\sin\Gamma(\mathscr{D}^2_{T_z,2}(\theta_i)-\mathscr{D}^2_{T_z,-2}(\theta_i))\},$$
$$Z^{2,0}_{T_z} = \sqrt{\tfrac{2}{3}}(\cos^2\Gamma-\tfrac{1}{2}\sin^2\Gamma)\mathscr{D}^2_{T_z,0}(\theta_i)-\tfrac{1}{2}\sin^2\Gamma(\mathscr{D}^2_{T_z,2}(\theta_i)+\mathscr{D}^2_{T_z,-2}(\theta_i)).$$

General property:

$$Z^{T,-m}_{T_z} = (-)^{T_z}Z^{T,m*}_{-T_z}.$$

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