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SOLUTION OF BOHR'S COLLECTIVE HAMILTONIAN

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Abstract: The Bohr collective Hamiltonian is diagonalized within the basis set of states corresponding to the quadrupole vibrator around the spherical equilibrium position. The dependence of the solution on the properties of the basis is discussed. Typical computer times used in the application of the method are given. The minimum number of spherical phonons that is needed in order to represent transition and well-deformed nuclei is also obtained.

1. Introduction

Many properties of nuclei may be understood in terms of the shell-model and the average quadrupole deformed field. This distorted field depends on the expectation value q_μ of the five components of the quadrupole operator. We can expect a very simple type of motion in which these expectation values change slowly with time, while the internal single-particle fields follow adiabatically.

However, instead of parametrizing the quadrupole distortion by the five q_μ, we may specify the deformations in an intrinsic frame and the orientation of this particular frame. For a quadrupole deformation, there are only two independent (and real) deformation parameters different from zero, namely the parameter β measuring the deformation independently of the system of reference and the parameter γ characterizing the departures from axial symmetry. The parameters β and γ are referred to as the intrinsic collective coordinates.

The Bohr-Hamiltonian [1]) depends on the intrinsic collective variables β, γ and of the Euler angles θ_i specifying the orientation of the intrinsic system. It can be easily solved in the limiting situation in which the motion corresponds to rotations for fixed values of β, and $\gamma = 0$. In such cases, collective states at somewhat higher energies are generated by vibrations of the system about the equilibrium shape (β and γ degrees of freedom).

In approximately spherical nuclei, the energies associated with the five collective degrees of freedom are about equal and one obtains a phonon-like spectrum.

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Although many nuclei closely follow the rotational pattern, even more cases cannot be fitted by either of the two simple solutions. Therefore, many attempts have been made in order to relax some of the requirements characterizing the rotational and vibrational descriptions. Examples of these attempts are, for instance, the rotations of non-axially symmetric nuclei [2]), the coupling of rotations and the β-degree of freedom [3]), the γ-independent potential energies [4, 5]), etc. However, the consequent improvements in the agreement with the empirical spectrum are not sufficiently general. The need to solve exactly Bohr's collective Hamiltonian thus becomes apparent. This solution has been carried out by Kumar and Baranger [6]) as a part of their program, which includes also the determination of the parameters of the collective Hamiltonian on the basis of the pairing plus quadrupole model.

In this paper we solve the same problem using a different method, having some advantages from the point of view of simplicity (and consequently of computing time). We diagonalize the Hamiltonian in the basis provided by the spherical quadrupole bosons. Sects. 2 and 3 treat the construction of the corresponding matrix elements. In sect. 4 we discuss the convergence of the solution as a function of the dimension of the basis and of the properties of the functions used in this basis.

With our method, we also obtain automatically the transformation connecting the vibrational and rotational wave functions.

2. Harmonic solution of the Bohr Hamiltonian

We review here some properties of the solution corresponding to the harmonic case. The classical Hamiltonian [1]) is

$$H_0 = \tfrac{1}{2}B\sum_\mu |\dot{\alpha}_\mu|^2 + \tfrac{1}{2}C\sum_\mu |\alpha_\mu|^2, \qquad \hbar\omega = \sqrt{\frac{C}{B}}, \tag{1}$$

where the variables α_μ transform under rotations like the components of a quadrupole tensor. For instance, they may be taken to be proportional to the quadrupole moment

$$\alpha_\mu = \frac{4\pi}{3AR_0^2}\sqrt{\frac{2C}{\hbar\omega}}\,Q_{2\mu}, \qquad Q_{2\mu} = \langle r^2 Y_{2\mu}\rangle. \tag{2}$$

The systems of coordinates corresponding to the principal axis is characterized by the fact that $\alpha'_1 = \alpha'_{-1} = \alpha'_2 - \alpha'_{-2} = 0$. The parameters α'_μ are related to the α_μ (which are defined in the lab frame of reference) by the relation

$$\alpha'_\mu = \sum_\nu D^2_{\mu\nu}(\theta_i)\alpha_\nu, \tag{3}$$

depending on the Euler angles θ_i.

It is customary to define the intrinsic deformation parameters β and γ [ref. [1])]

$$\beta^2 = \sum_\mu |\alpha_\mu|^2 = \alpha_0'^2 + 2\alpha_2'^2,$$

$$\gamma = \operatorname{arctg} \frac{\sqrt{2}\alpha_2'}{\alpha_0'} \tag{4}$$

By expressing the kinetic energy in terms of β, γ and θ_i, and using the Pauli prescription in the quantization procedure, Bohr [1]) obtains the Hamiltonian

$$H_0 = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin^2 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4}\sum_{x=1}^{x=3}\frac{I_x^2}{\sin^2(\gamma - \frac{2}{3}\pi x)}\right] + \tfrac{1}{2}C\beta^2, \tag{5}$$

where I_x are the Cartesian components of the angular momentum operator along the principal axis.

The solutions corresponding to the Hamiltonian (5) have energies

$$E_{n,\lambda} = (2n + \lambda + \tfrac{5}{2})\hbar\omega, \tag{6}$$

and the wave functions may be written [4]) as a product

$$\Phi_{n,\lambda,I,M}(\beta, \gamma, \theta_i) = u_{n,\lambda}(\beta)\psi_{\lambda,I,M}(\gamma, \theta_i), \tag{7}$$

where the radial wave function $u_{n,\lambda}(\beta)$ is given by

$$u_{n,\lambda}(\beta) = \sqrt{2}A_{n,\lambda}\left(\frac{C}{\hbar\omega}\right)^{5/4}\rho^\lambda L_n^{\lambda+\frac{3}{2}}(\rho^2)\mathrm{e}^{-\frac{1}{2}\rho^2},$$

$$\rho = \sqrt{\frac{C}{\hbar\omega}}\,\beta, \qquad A_{n,\lambda} = \left[\frac{n!}{\Gamma^3(n+\lambda+\frac{5}{2})}\right]^{\frac{1}{2}} \tag{8}$$

and the $L_n^{\lambda+\frac{3}{2}}(\rho^2)$ are Laguerre polynomials [7]). The "spherical harmonics" corresponding to our five-dimensional space depend on the four angles γ, θ_i

$$\psi_{\lambda,I,M}(\gamma, \theta_i) = \sum_i w_i^{\lambda,I}(\cos 3\gamma)\sum_{K\geqq 0}\chi_{i,K}^I(\gamma)\left[\frac{2I+1}{8\pi^2(1+\delta_{K,0})}\right]^{\frac{1}{2}} \times \{D_{MK}^I(\theta_i) + (-)^I D_{M-K}^I(\theta_i)\}. \tag{9}$$

The functions $w_i^{\lambda,I}(\cos 3\gamma)$ are polynomials containing either even or odd powers of $\cos 3\gamma$. Recurrence relations for the coefficients are given in ref. [4]). The functions $\chi_{i,K}^I(\gamma)$ are also given explicitly in refs. [4, 6]) for $I \leqq 6$. There are as many $\chi_{i,K}^I(\gamma)$ as possible values of K in eq. (9) $[\frac{1}{4}(2I+1+(-)^I 3)]$. The second summation in eq. (9) is restricted to even values of K and $K = 0$ does not appear for odd values of I. The

quantum number λ may be interpreted [9]) as the seniority †. The functions $\psi_{\lambda,I,M}(\gamma,\theta_i)$ are independent of the quantum number n corresponding to the β-motion. Each time that n is increased by one we add an extra pair of phonons coupled to angular momentum zero, so for each λ we can construct a "β-band".

The volume element is written [1]) as

$$\mathrm{d}v = \beta^4|\sin 3\gamma|\sin\theta_2\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 \tag{10}$$

and the limits of integration are $0<\beta<\infty$, $0<\gamma<2\pi$, $0<\theta_1<2\pi$, $0<\theta_2<\pi$ and $0<\theta_3<2\pi$.

For $I<6$, the quantum numbers n, λ, I, M completely determine the wave function.

3. Anharmonic Hamiltonian

3.1. VELOCITY INDEPENDENT TERMS

As the Hamiltonian must be invariant with respect to rotations, it is possible to express the potential energy $V(\beta,\gamma)$ as a linear combination of the functions $\Phi_{n,\lambda,0,0}(\beta,\gamma)$, i.e., it may be expanded in powers of β^2 and $\beta^3\cos 3\gamma$ [refs. [4,6])]. Therefore, we need to calculate matrix elements of operators $\beta^k\cos^S 3\gamma$ (k, S = integers). The β-part of the integral yields the value (p. 785 of ref. [7]))

$$\langle n\lambda|\beta^k|n'\lambda'\rangle = \left(\frac{\hbar\omega}{C}\right)^{\frac{1}{4}k}\left\{A_{n,\lambda}A_{n',\lambda'}\int_0^\infty x^{(\lambda+\lambda'+k+3)/2}\mathrm{e}^{-x}L_n^{\lambda+\frac{3}{2}}(x)L_{n'}^{\lambda'+\frac{3}{2}}(x)\mathrm{d}x\right\}$$

$$= (-)^{n+n'}\left(\frac{\hbar\omega}{C}\right)^{\frac{1}{4}k}A_{n,\lambda}A_{n',\lambda'}\Gamma(n+\lambda+\tfrac{5}{2})\Gamma(n'+\lambda'+\tfrac{5}{2})\left(\frac{\lambda'+k-\lambda}{2}\right)!\left(\frac{\lambda+k-\lambda'}{2}\right)!$$

$$\times\sum_\sigma\frac{\Gamma(n+\sigma+\frac{1}{2}(\lambda+\lambda'+k+5))}{\sigma!(n-\sigma)!(n'-\sigma)!(\sigma+\frac{1}{2}(\lambda'+k-\lambda)-n)!(\sigma+\frac{1}{2}(\lambda+k-\lambda')-n')!}, \tag{11}$$

if $\lambda+\lambda'+k$ is even.

It must be noted that for all the terms appearing in the Hamiltonian $\lambda+\lambda'+k$ is even. The selection rules for the γ-integrals indicate that the matrix elements of $\cos^S 3\gamma$ are different from zero only if $\lambda+\lambda'$ has the same parity as S. As k is of the form $2m+2S$, $\lambda+\lambda'+k$ is always even.

After performing the integral over the Euler angles, the γ-dependent part of the integrand in the matrix elements of $\cos^k 3\gamma$ can be written as a polynomial $h_I^{\lambda\lambda'}$ in $\cos 3\gamma$. For $I\leqq 4$, these polynomials are

$$h_0^{\lambda\lambda'}(\cos 3\gamma) = ww',$$

$$h_2^{\lambda\lambda'}(\cos 3\gamma) = ww'+vv'+(wv'+w'v)\cos 3\gamma,$$

$$h_4^{\lambda\lambda'}(\cos 3\gamma) = 36ww'+(15+21\cos^2 3\gamma)vv'+60(1-\cos^2 3\gamma)uu'$$
$$+30(1-\cos^2 3\gamma)(wu'+w'u)-36\cos 3\gamma(wv'+w'v), \tag{12}$$

where the polynomials w, v, u are given explicitly in ref. [4]).

† Of course, the same happens in the three-dimensional harmonic oscillator, where the orbital angular momentum also has the meaning of a seniority.

Therefore, the matrix elements

$$\langle\psi_{\lambda,I,M}|\cos^k 3\gamma|\psi_{\lambda',I,M}\rangle = \int_0^{2\pi} h_I^{\lambda\lambda'}(\cos 3\gamma)\cos^k 3\gamma|\sin 3\gamma|\mathrm{d}\gamma \tag{13}$$

yield a sum of integrals of the type

$$\int_0^{2\pi}\cos^S 3\gamma|\sin 3\gamma|\mathrm{d}\gamma = \begin{cases} 0 & \text{if } S \text{ is odd,} \\ 4\int_0^1 x^S\,\mathrm{d}x = \dfrac{4}{1+S} & \text{if } S \text{ is even.} \end{cases} \tag{14}$$

Some of the matrix elements are given in the appendix.

3.2. VELOCITY DEPENDENT TERMS

We assume that the motion is adiabatic (i.e. we neglect powers of $\dot{\alpha}_\mu$ higher than the second). As the kinetic energy must be hermitian and invariant under rotation and time-reversal, one can write

$$T = \frac{1}{2B}\sum_\mu |P_\mu|^2 + B^{(3)}\{\alpha\{PP\}^2\}^0 + \tfrac{1}{2}\sum_l B_l^{(4)}[\{\{PP\}^l\{\alpha\alpha\}^l\}^0 + \{\{\alpha\alpha\}^l\{PP\}^l\}^0] + \text{higher order terms,} \tag{15}$$

where the quantities α_μ are given in eq. (2) and the P_μ are their conjugate momenta. Their relation with the boson creation and annihilation operators is given by

$$\alpha_\mu = \sqrt{\frac{\hbar\omega}{2C}}(b_\mu^+ + (-)^\mu b_{-\mu}),$$

$$P_\mu = i\sqrt{\frac{C}{2\hbar\omega}}(b_\mu^+ - (-)^\mu b_{-\mu}). \tag{16}$$

The matrix elements of the third order term in eq. (15) can be evaluated by comparing its expansion in creation and annihilation operators with the corresponding expansion [11]) of $\beta^3\cos 3\gamma$

$$\frac{\langle n\lambda IM|\{\alpha\{PP\}^2\}^0|n'\lambda'I'M'\rangle}{\langle n\lambda IM|\beta^3\cos 3\gamma|n'\lambda'I'M'\rangle} = \sqrt{\tfrac{2}{35}}CB(\tfrac{1}{3}\pm\tfrac{2}{3}) \begin{cases} + & \text{if } 2n+\lambda = 2n'+\lambda'\pm 3, \\ - & \text{if } 2n+\lambda = 2n'+\lambda'\pm 1. \end{cases} \tag{17}$$

Therefore, the contributions from the term $B^{(3)}$ can be obtained by multiplying by an appropriate constant the matrix elements of $\beta^3\cos 3\gamma$.

Because of the symmetrization of the fourth order terms, it is easily seen by replacing eqs. (16) in (15) that these terms in the kinetic energy do not connect states differing in two in the number of phonons. Moreover, the matrix elements between states differing by four phonons are identical with those of the operator

$$-\sum_l B_l^{(4)}\{\{\alpha\alpha\}^l\{\alpha\alpha\}^l\}^0 = K_1\beta^4,$$

$$K_1 = \tfrac{1}{35}(C/\hbar\omega)^2[2\sum_l B_l^{(4)}(2l+1)^{\frac{1}{2}} + 5B_0^{(4)}]. \tag{18}$$

Finally, the matrix elements between states with the same number of phonons are those of the operator

$$-\sum_l B_l^{(4)}\{\{b^+b^+\}^l\{bb\}^l\}^0+K_1\beta^4-\tfrac{5}{2}K_2-K_2\sqrt{5}\{b^+b\}^0,$$
$$K_2=\tfrac{2}{5}\sum_l B_l^{(4)}(2l+1)^{\frac{1}{2}}. \tag{19}$$

In order to evaluate expression (19) we need to know the matrix elements of the operators $\{\{b^+b^+\}^l\{bb\}^l\}^0$. It is easy to verify that the three of them commute with the harmonic Hamiltonian. This suggests that they may be diagonal within our representation.

The operators corresponding to the square of the number of phonons N^2 and of the angular momentum I^2 are given by

$$N^2=(\sqrt{5}\{b^+b\}^0)^2=\sqrt{5}\{b^+b\}^0+\sum_l(2l+1)^{\frac{1}{2}}\{\{b^+b^+\}^l\{bb\}^l\}^0, \tag{20}$$

$$I^2=-\sqrt{3}\{\boldsymbol{II}\}^0=-10\sqrt{3}\{\{b^+b\}^1\{b^+b\}^1\}^0$$
$$=6\sqrt{5}\{b^+b\}^0+30\sum_l(2l+1)^{\frac{1}{2}}\begin{Bmatrix}l & 2 & 2\\ 1 & 2 & 2\end{Bmatrix}\{\{b^+b^+\}^l\{bb\}^l\}^0. \tag{21}$$

The diagonal component of the operator β^4 is

$$\beta^4_{\text{diag}}=5\{\alpha\alpha\}^0\{\alpha\alpha\}^0_{\text{diag}}$$
$$=\left(\frac{\hbar\omega}{C}\right)^2\left[\frac{35}{4}+\frac{7}{\sqrt{5}}\{b^+b\}^0\right]+\sum_l((2l+1)^{\frac{1}{2}}+\tfrac{5}{2}\delta_{l0})\{\{b^+b^+\}^l\{bb\}^l\}^0. \tag{22}$$

The matrix elements of β^4_{diag} are obtained from (11)

$$\langle n\lambda IM|\beta^4|n\lambda IM\rangle=\left(\frac{\hbar\omega}{C}\right)^2(6n^2+15n+\lambda^2+6\lambda+6\lambda n+\tfrac{35}{4}). \tag{23}$$

Using the eigenvalues of N^2, I^2 plus eqs. (20), (21), (22) and (23), we obtain the matrix elements corresponding to $l=0$, 2 and 4

$$\langle n\lambda IM|\{b^+b^+\}^0\{bb\}^0|n\lambda IM\rangle=\tfrac{4}{5}n(n+\lambda+\tfrac{3}{2}),$$

$$\langle n\lambda IM|\{\{b^+b^+\}^2\{bb\}^2\}^0|n\lambda IM\rangle=\frac{1}{7\sqrt{5}}[8n^2+4\lambda^2+8n\lambda-8n+2\lambda-I(I+1)], \tag{24}$$

$$\langle n\lambda IM|\{\{b^+b^+\}^4\{bb\}^4\}^0|n\lambda IM\rangle=\tfrac{1}{105}[72n^2+15\lambda^2+72n\lambda-72n-45\lambda+5I(I+1)].$$

Therefore, according to (18), (19) and (24), the matrix elements of the fourth order anharmonicities in the kinetic energy are also obtained by multiplying by appropriate constants the corresponding matrix elements of the potential energy ($\langle i|\beta^4|i'\rangle$). In addition, we must add diagonal terms depending only on the quantum numbers n, λ and I.

5. Convergence of the method

Once the matrix elements of the anharmonic Hamiltonian are calculated according to the method described in the previous section, we proceed to diagonalize the corresponding matrix. We want to investigate in this section how the results depend on the dimensions of our basic set and on the parameters of the harmonic oscillator basis.

It is obvious that the necessary number of phonons will depend on the anharmonicity of the Hamiltonian. Therefore, we shall specifically consider the dependence of the final solution with the number of phonons for a potential that is able to stabilize the system at $\beta = 0$ and values of β close to the values appearing in well-deformed nuclei. Because the eigenstates of the phonon basis do not have a definite value of K [eqs.(7) and (9)] as axially symmetric deformed nuclei do, our resultant wave function should consist of at least several phonons in order to cancel the components with K different than the main one, in the whole of the β-γ plane.

The expectation value of β^2 in a state with a particular number of phonons is $(N+\frac{5}{2})/B\hbar\omega$ and, therefore if the values of $\hbar\omega$ are too large, we should need many phonons in order to have wave functions significantly different from zero in the region corresponding to the equilibrium deformation. On the other hand, if $\hbar\omega$ is too small, the number of phonons should again be large in order to obtain sufficient cancellations in the wave function for deformations far greater than the equilibrium deformation.

We may apply three criteria in order to check the accuracy of the method. The first one is to insure that sufficiently small changes appear in our solutions when either of the two parameters, $\hbar\omega$ and N, is changed.

The second criterion is to compare our results with those obtained by Baranger and Kumar [6]). In their case also, two parameters may be varied, namely the radius β_m of the circumference in the β, γ plane, within which the equation is solved, and the fineness of the mesh in which this portion of the plane is divided. No direct correspondence appears between these parameters and ours and, therefore, we may consider the two methods as being complementary approximations.

The mth wave function with angular momentum I is obtained as a linear combination

$$|^mI, M\rangle = \sum_{n,\lambda} a^{m,I}_{n,\lambda} \Phi_{n,\lambda,I,M}(\beta, \gamma, \theta_i) \tag{25}$$

and thus it represents a wave packet constructed with our basic set (7). The average number of phonons $n(^mI)$ and the width $\Delta(^mI)$ of the wave packet can be written

$$n(^mI) = \sum_{n,\lambda} |a^{m,I}_{n,\lambda}|^2 (2n+\lambda), \tag{26}$$

$$\Delta(^mI) = \{\sum_{n,\lambda} |a^{m,I}_{n,\lambda}|^2 (2n+\lambda-n(^mI))^2\}^{\frac{1}{2}}. \tag{27}$$

In order to have a reliable solution the two quantities (25) and (26) should not vary significantly if the number of phonons is changed by one unit. In this way, we insure that the omitted phonon did not have an important amplitude in the wave function.

In order to compare our results with those of refs. [6,8]) we use the potential developed by Kumar [8]) for ^{154}Sm

$$V(\beta,\gamma) = \tfrac{1}{2}C\beta^2 - f\beta^3\cos 3\gamma + \left(G_0 + G_1\left(\frac{\beta}{a}\right)^3\cos 3\gamma + G_2\left(\frac{\beta}{a}\right)^6\cos^2 3\gamma\right)e^{-\beta^2/a^2}, \quad (28)$$

with $C = 77$ MeV, $f = 9.7$ MeV, $a = 0.3$, $G_0 = 6.7$ MeV, $G_1 = -5.5$ MeV and $G_2 = -1.3$ MeV. The value of B is 127 MeV^{-1} †.

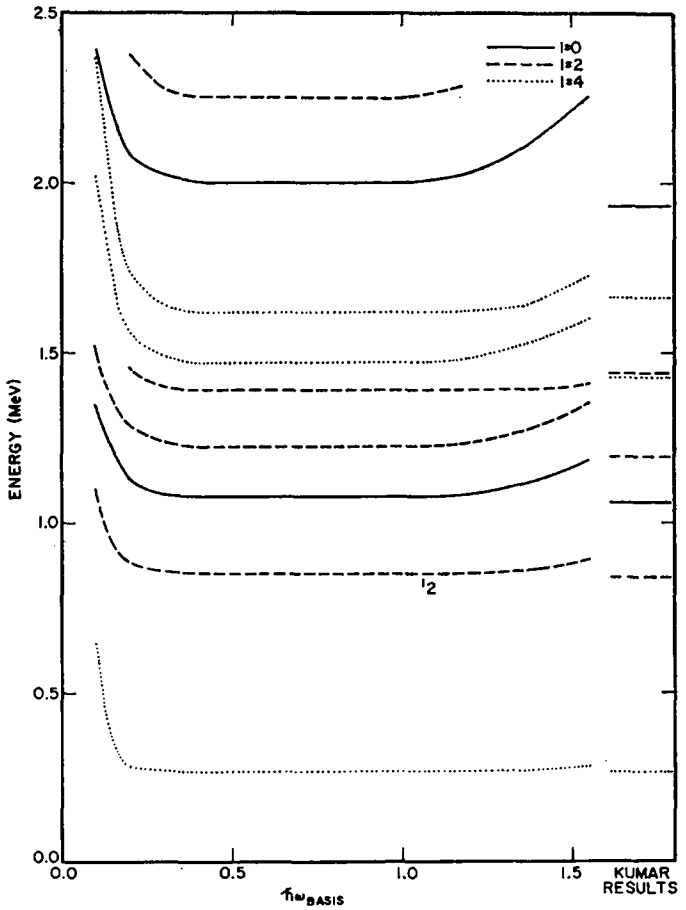

Fig. 1. Energies as functions of $\hbar\omega$ and comparison with Kumar's [8]) results using the potential (28) and 20 phonons. The energy of the ^{12} state has been multiplied by 10.

† The matrix elements of the Gaussian factor in (28) can be easily calculated within our scheme. We notice that in the same way that the matrix elements of β^k are calculated by adding integrals of the form $\int_0^\infty e^{-\rho^2}\rho^S d\rho$ according to (11), the matrix elements of $\beta^k e^{-\beta^2/a^2}$ may be also calculated by adding terms of the form $\int_0^\infty \rho^S e^{-\rho^2(1+(\hbar\omega/Ca^2))}\, d\rho$.

Fig. 1 shows the energies resultant from our diagonalization including up to 20 phonons, and corresponding to different values of $\hbar\omega$. Fig. 2 shows similar results for the transition rates. We notice that energies and transition rates are almost independent of $\hbar\omega$ in the interval 0.3 MeV $\leqq \hbar\omega \leqq$ 1.2 MeV. Within this interval, the devia-

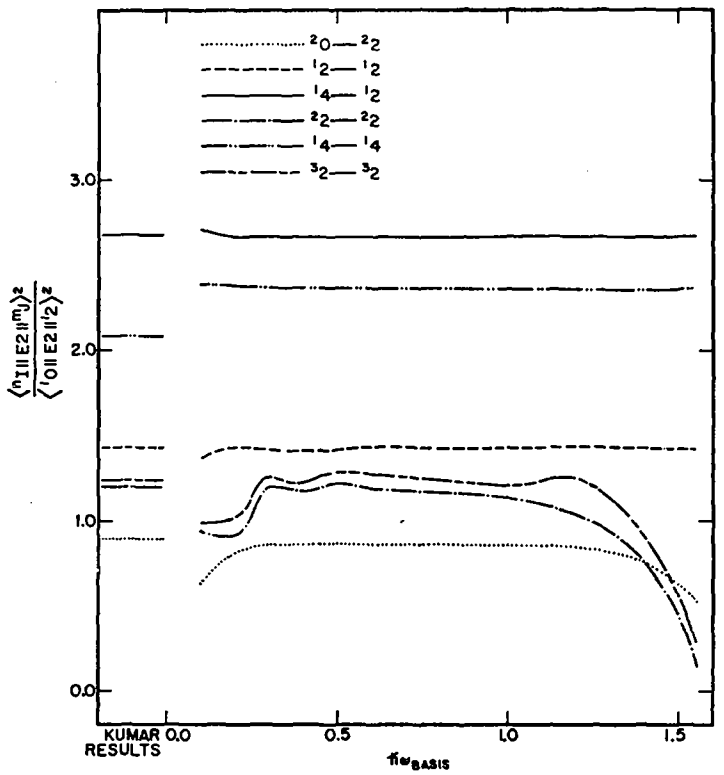

Fig. 2. E2 ratios as function of $\hbar\omega$ and comparison with Kumar's [8]) results using the potential (28) and 20 phonons.

TABLE 1

The average number of phonons $n(^mI)$ and the dispersion $\Delta(^mI)$ for different frequencies of the harmonic oscillator basis and 20 phonons

$\hbar\omega$	$n(^1 0)$	$\Delta(^1 0)$	$n(^2 0)$	$\Delta(^2 0)$	$n(^1 2)$	$\Delta(^1 2)$	$n(^2 2)$	$\Delta(^2 2)$	$n(^1 4)$	$\Delta(^1 4)$
1.0	6.04	2.59	5.81	4.52	6.33	2.71	6.62	4.30	6.84	2.44
0.8	4.79	2.19	4.70	3.79	5.03	2.10	5.52	3.56	5.52	2.01
0.7	4.18	2.04	4.20	3.47	4.44	1.96	5.03	3.22	4.94	1.85
0.6	3.62	1.95	3.90	3.22	3.90	1.87	4.66	2.94	4.41	1.76
0.5	3.18	1.98	3.55	3.08	3.45	1.90	4.43	2.77	4.02	1.79
0.4	2.88	2.20	3.53	3.12	3.19	2.11	4.43	2.80	3.83	2.02
0.3	2.86	2.71	3.83	3.46	3.22	2.60	4.84	3.12	4.00	2.52
0.2	3.29	3.52	4.54	3.96	3.83	3.39	5.80	3.67	4.81	3.22
0.1	4.20	4.62	5.44	3.00	5.11	4.02	9.03	3.50	6.93	3.59

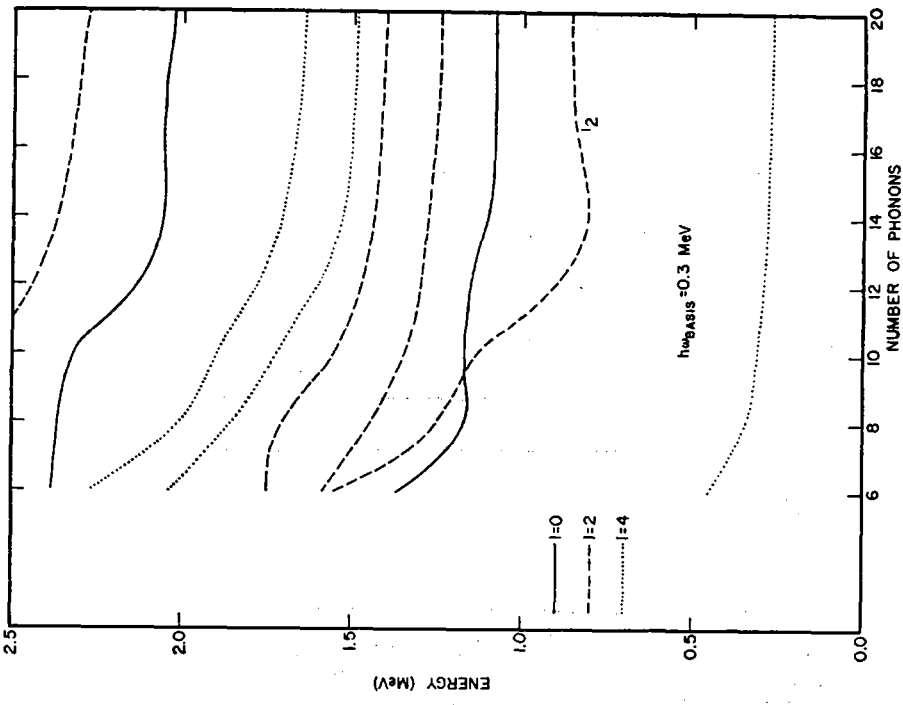

Fig. 3b. Energies obtained for $\hbar\omega = 0.3$ MeV using Kumar's [8]) potential, as a function of the number of phonons included. The energy of the I_2 state has been multiplied by 10.

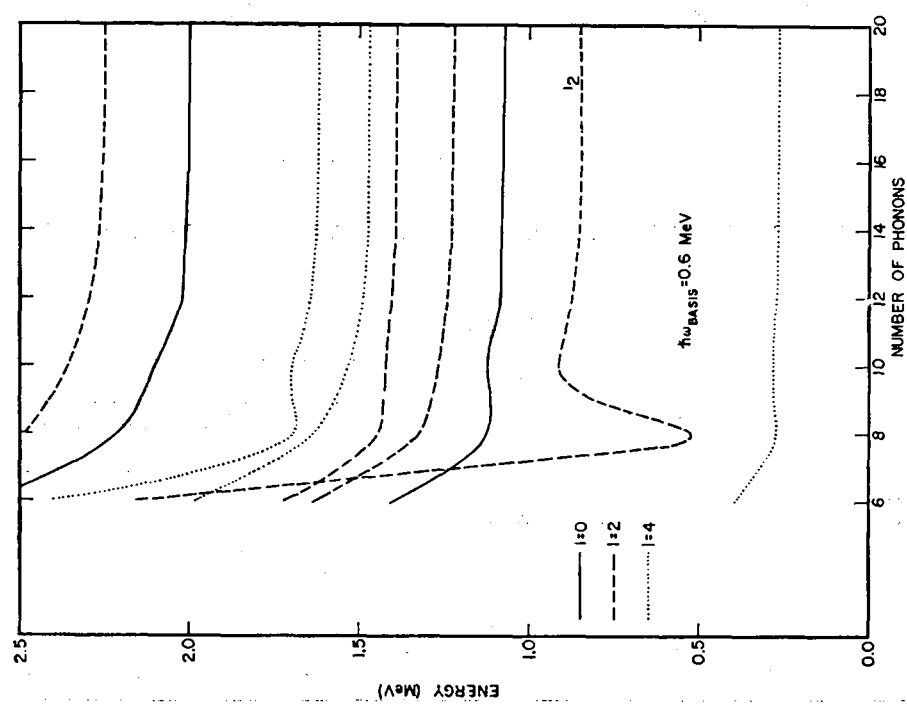

Fig. 3a. Energies obtainced for $\hbar\omega = 0.6$ MeV using Kumar's [8]) potential as a function of the number of phonons included. The energy of the I_2 state has been multiplied by 10.

tions from the average are less than 3 % for the energies and less than 3 % for the transition rates in the cases that are represented.

The comparison with the results of refs. [6,8]) is made on the right- and left-hand side of figs. 1 and 2, respectively, and the agreement is very good in general. The only minor exceptions are the inversion in the order of the closely spaced $^{3}2$ and $^{2}4$ states, and the static quadrupole moment of the $^{1}4$ state.

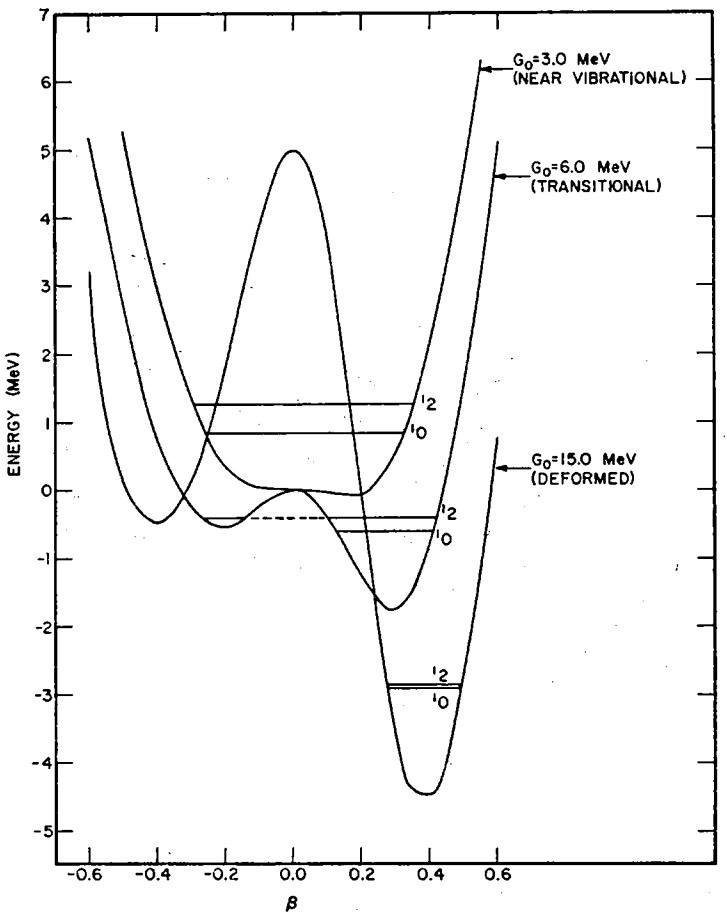

Fig. 4. Potential well for three G_0. The vertical scale has been displaced to allow the reproduction of the potentials in the same figure.

The values of the mean number of phonons (26) and of the dispersion (27) are listed in table 1 for the levels $^{1}0$, $^{2}0$, $^{1}2$, $^{2}2$ and $^{1}4$. In the interval 0.3 MeV $\leqq \hbar\omega \leqq$ 0.8 MeV they are reasonably stable and they indicate that a smaller basis could have been used as well.

Therefore, we repeated the calculations for different numbers of phonons, in the optimum case of $\hbar\omega = 0.6$ MeV and in the borderline case $\hbar\omega = 0.3$ MeV. The results are represented in figs. 3a and 3b. While we need at least 12 phonons in the optimum case, about 16-18 phonons are necessary in the borderline case. This last result is clearly consistent with the fact that the spectrum given by Kumar, or the one obtained for $\hbar\omega = 0.6$ MeV (fig. 1) is barely reproduced with 20 phonons and $\hbar\omega = 0.3$

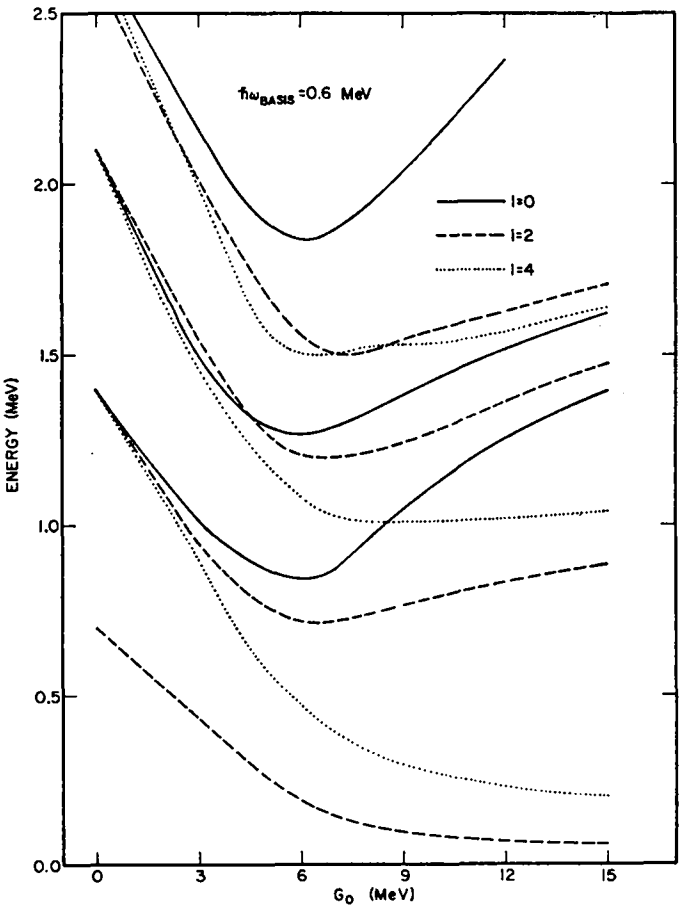

Fig. 5. Excitation spectrum as function of G_0 with $G_1 = -\frac{1}{3}G_0$, $G_2 = f = 0$, $a = 0.3$ and $C = 62$ MeV, for $\hbar\omega = 0.6$ MeV.

MeV. The mean values (26) and widths (27) vary very little if the numbers of phonons is greater or equal than 12 (if $\hbar\omega = 0.6$ MeV) and they are not nearly as constant for $\hbar\omega = 0.3$ MeV.

The potential (28) implies an energy difference of about 3.75 MeV between the minimum at $\beta = 0.35$ and the value at $\beta = 0$. Therefore, it is rather effective in stabilizing the system at the equilibrium deformation. But we are also interested in

studying transitional nuclei. Obviously, our representation yields exact results, with a very small number of phonons, in the limit of vibrations around the spherical equilibrium position. Therefore, we may expect that the necessary number of phonons decreases continuously when going from deformed to spherical nuclei.

In order to study the convergence for transition nuclei, we use a potential similar to (2) with $C = 62$ MeV, $f = G_2 = 0$ and $G_1 = -\frac{1}{3}G_0$.

$$V(\beta, \gamma) = \tfrac{1}{2}C\beta^2 + G_0\left(1 - \frac{1}{3}\left(\frac{\beta}{a}\right)^3 \cos 3\gamma\right) e^{-\beta^2/a^2}. \tag{29}$$

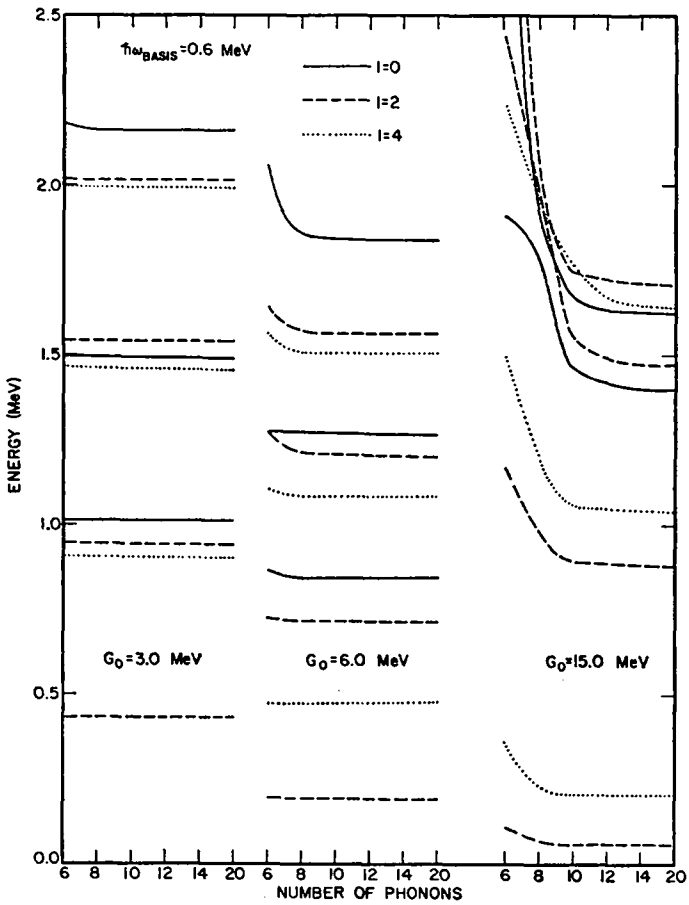

Fig. 6. Energies as a function of the number of phonons for $\hbar\omega = 0.6$ MeV, for three G_0.

Again $B = 127$ MeV^{-1} and G_0 is allowed to vary between 0 and 15 MeV (fig. 4). Fig. 5 shows the excitation spectrum as a function of G_0. In particular, we see that the slope of those levels which eventually become members of the ground state rotational band is always negative, and therefore the ratio between their energies (or the

deviation from the $I(I+1)$ law) is not a very sensitive test of how far we are from the pure vibrational or pure rotational descriptions. The levels that will constitute the β- and γ-vibrational states show a characteristic minimum in the region where the random-phase approximation would predict a zero energy root, i.e. a phase transition. In the rotational limit both energy considerations and transition rates allow the identification of the ^{2}0 and ^{3}0 levels as the β-vibrational and two γ-vibrations respectively.

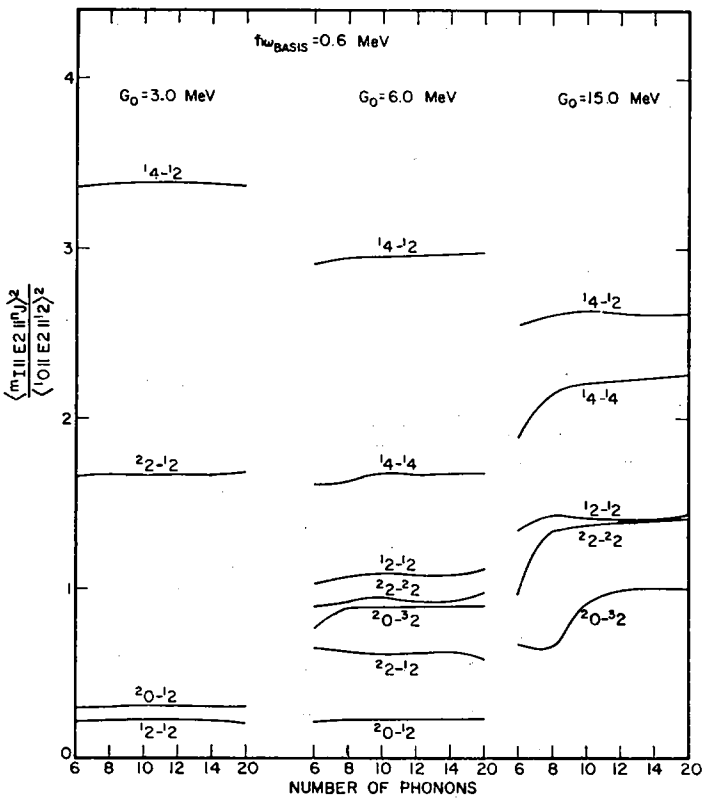

Fig. 7. E2 ratios as a function of the number of phonons for $\hbar\omega$ = 0.6 MeV, for three G_0.

Figs. 6 and 7 show energies and E2 matrix elements as a function of the number of bosons for $\hbar\omega$ = 0.6 MeV. In particular, the results on the static quadrupole moment of the ^{1}2 state suggest that we need an appreciable stabilization around the deformed equilibrium position in order to obtain a ratio $\langle {}^1 2||E2||{}^1 2\rangle / \langle {}^1 2||E2||{}^1 0\rangle$ characteristic of well-deformed nuclei. This somewhat contradicts other results that have recently appeared in the literature [10, 11]). The discrepancy is probably due to our parametrization of the potential energy surface (29) implying a definite relationship between the γ-stability and the depth of the deformed minimum. But we are not suggesting that (29) is a realistic representation of the potential for transition nuclei. We are simply using (29) in order to study the convergence of the method as a function of the num-

ber of phonons when the stability around the equilibrium deformation increases. Table 3 includes the mean number of phonons (26), the dispersion (27) and the minimum number of phonons n_{min} (i.e. the number of phonons such that the results do not change significantly when using a larger basis). We see that these quantities vary smoothly with G_0 and the advantage of $\hbar\omega = 0.6$ MeV with respect to the other two values.

TABLE 2

No. of phonons	$\hbar\omega$	$n(^1 0)$	$\Delta(^1 0)$	$n(^2 0)$	$\Delta(^2 0)$	$n(^1 2)$	$\Delta(^1 2)$	$n(^2 2)$	$\Delta(^2 2)$	$n(^1 4)$	$\Delta(^1 4)$
20	0.6	3.62	1.95	3.90	3.22	3.90	1.87	4.66	2.94	4.41	1.76
18	0.6	3.60	1.95	3.93	3.22	3.90	1.87	4.64	2.94	4.41	1.76
16	0.6	3.62	1.95	3.89	3.21	3.92	1.87	4.60	2.93	4.41	1.75
14	0.6	3.60	1.94	3.85	3.18	3.89	1.85	4.62	2.91	4.41	1.74
12	0.6	3.59	1.93	3.83	3.17	3.86	1.83	4.51	2.84	4.39	1.72
10	0.6	3.55	1.87	3.44	3.02	3.80	1.77	4.38	2.77	4.30	1.64
8	0.6	3.26	1.72	3.24	2.88	3.64	1.68	3.80	2.41	4.09	1.50
6	0.6	2.96	1.64	1.76	2.34	3.04	1.41	2.78	1.61	3.59	1.28
20	0.3	2.86	2.71	3.80	3.46	3.21	2.60	4.84	3.12	4.00	2.52
18	0.3	2.79	2.64	3.79	3.41	3.21	2.55	4.65	3.01	3.92	2.43
16	0.3	2.72	2.53	3.70	3.31	3.13	2.47	4.55	2.81	3.92	2.43
14	0.3	2.63	2.42	3.49	3.08	3.06	2.36	4.38	2.64	3.75	2.17
12	0.3	2.50	2.32	3.07	2.77	2.90	2.17	4.24	2.47	3.61	2.00
10	0.3	2.35	2.17	2.78	2.64	2.67	1.93	3.85	2.26	3.43	1.80
8	0.3	2.06	1.92	2.62	2.44	2.42	1.70	3.29	1.95	3.16	1.48
6	0.3	1.68	1.66	1.83	1.79	2.11	1.43	3.09	1.30	2.82	1.03

The average number of phonons $n(^m I)$ and the dispersion $\Delta(^m I)$ for two frequencies ($\hbar\omega = 0.6$ MeV and $\hbar\omega = 0.3$ MeV) and different number of phonons.

TABLE 3

The average number of phonons $n(^m I)$ and the dispersion $\Delta(^m I)$ for different values of the constant G_0 in (29) and a basis of 20 phonons

G_0	$\hbar\omega$	$n(^1 0)$	$\Delta(^1 0)$	$n(^2 0)$	$\Delta(^2 0)$	$n(^1 2)$	$\Delta(^1 2)$	$n(^2 2)$	$\Delta(^2 2)$	$n(^1 4)$	$\Delta(^2 4)$	n_{min}
3.0	0.4	0.04	0.41	2.28	0.72	1.12	0.52	2.16	0.67	2.23	0.66	8
3.0	0.6	0.14	0.53	2.11	0.82	1.14	0.51	2.06	0.49	2.14	0.48	6
3.0	0.8	0.47	0.94	2.43	1.58	1.53	0.98	2.44	0.99	2.55	0.99	8
6.0	0.4	0.76	1.25	2.28	1.74	1.58	1.09	2.14	1.11	2.48	0.99	10
6.0	0.6	1.41	1.41	2.54	2.16	2.18	1.25	2.57	1.24	2.96	1.12	8
6.0	0.8	2.27	1.81	3.25	2.84	3.06	1.69	3.37	1.70	3.84	1.59	10
15.0	0.4	3.42	1.81	5.07	2.94	3.61	1.77	4.59	1.95	4.03	1.69	14
15.0	0.6	4.78	1.75	6.06	3.05	4.94	1.72	5.72	1.78	5.27	1.65	12
15.0	0.8	6.47	2.11	7.54	3.67	6.60	2.08	7.37	2.13	6.92	2.03	14

The last column indicates the minimum number of phonons which is needed.

The diagonalization including up to 20 phonons corresponds to matrices of order 44, 77 and 100, for the $I = 0$, 2 and 4 states, respectively. The diagonalization is performed using the subroutine MEERA, the same one used in refs. [6,8]). In the Univac 1108, typical machine times used in order to obtain five $I = 0$ levels, five $I = 2$ levels and three $I = 4$ levels, are 15 second in the case of 20 phonons and 3 seconds in the case of 14 phonons, from the moment we input the parameters of $V(\beta, \gamma)$.

5. Conclusions

The method of solving the Bohr's collective Hamiltonian by diagonalizing it in the basic set of states corresponding to spherical bosons appears to be simpler and to yield results with similar accuracy as other previously developed procedures [6]). In order for this to be true, we must choose adequately the frequency of the harmonic basis. A minimum number of phonons is also required in order to describe rotational and transitional nuclei, depending on the stability around the deformed equilibrium position. The fact that this number is sufficiently small implies that the spherical basis probably constitutes the most convenient set of states that may be used in order to study the development of the strong coupling approximation. In addition, the present method may be easily generalized to cases in which the coupling with other collective degrees of freedom, such as pairing vibrations [12]), becomes important, simply by using as basic set the product of the two Hilbert spaces corresponding to quadrupole and pairing bosons.

We are very thankful to Dr. K. Kumar who provided us with a copy of his code MEERA based on the diagonalization of large matrices. Discussions with Professors M. Baranger, B. Bayman, R. Broglia, R. Sorensen and Dr. R. Perazzo are gratefully acknowledged.

Appendix

1) $\underline{I = 0;\ \lambda = 3m}$

$$\langle \lambda | \cos 3\gamma | \lambda + 3 \rangle = \frac{m+1}{\sqrt{(2m+1)(2m+3)}},$$

$$\langle \lambda | \cos^2 3\gamma | \lambda \rangle = \frac{2m^2+2m-1}{(2m-1)(2m+3)},$$

$$\langle \lambda | \cos^2 3\gamma | \lambda + 6 \rangle = \frac{(m+1)(m+2)}{(2m+3)\sqrt{(2m+1)(2m+5)}}.$$

2) $I = 2$; $\lambda = 3m+\delta$ with $\delta = 1, 2$

$$\langle\lambda|\cos 3\gamma|\lambda+1\rangle = \frac{1}{(2m+1)(2m+3)},$$

$$\langle\lambda|\cos 3\gamma|\lambda+3\rangle = -\frac{\sqrt{(m+1)(m+2)}}{(2m+3)},$$

$$\langle\lambda|\cos^2 3\gamma|\lambda\rangle = \frac{2m^2+4m+1}{(2m+1)(2m+3)},$$

$$\langle\lambda|\cos^2 3\gamma|\lambda+\tfrac{2}{4}\rangle = \begin{cases} 0 \\ -2\dfrac{\sqrt{(m+1)(m+2)}}{(2m+1)(2m+3)(2m+5)} \end{cases}$$

$$\langle\lambda|\cos^2 3\gamma|\lambda+6\rangle = \frac{(m+2)\sqrt{(m+1)(m+3)}}{(2m+3)(2m+5)}.$$

3) $I = 4$

$\lambda = 3m+2$

$$\langle\lambda|\cos 3\gamma|\lambda+1\rangle = -\left[\frac{210}{(2\lambda+1)(2\lambda+3)(2\lambda+7)(2m+3)}\right]^{\frac{1}{2}},$$

$$\langle\lambda|\cos 3\gamma|\lambda+3\rangle = \left[\frac{(2\lambda+1)(m+1)(m+3)}{(2\lambda+7)(2m+3)^2}\right]^{\frac{1}{2}}.$$

$\lambda = 3m+3$

$$\langle\lambda|\cos 3\gamma|\lambda+1\rangle = -\left[\frac{210}{(2\lambda+1)(2\lambda+5)(2\lambda+7)(2m+3)}\right]^{\frac{1}{2}},$$

$$\langle\lambda|\cos 3\gamma|\lambda+3\rangle = \left[\frac{(2\lambda+1)(2\lambda+11)(m+1)(m+3)}{(2\lambda+5)(2\lambda+7)(2m+3)(2m+5)}\right]^{\frac{1}{2}}.$$

$\lambda = 3m+4$

$$\langle\lambda|\cos 3\gamma|\lambda+1\rangle = 8\left[\frac{(m+1)(m+3)}{(2\lambda+3)(2\lambda+5)(2m+5)^2(2m+3)^2}\right]^{\frac{1}{2}},$$

$$\langle\lambda|\cos 3\gamma|\lambda+3\rangle = \left[\frac{(2\lambda+11)(m+1)(m+3)}{(2\lambda+5)(2m+5)^2}\right]^{\frac{1}{2}}.$$

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