WILSON LOOP LANGEVIN EQUATIONS FOR U(1) LATTICE GAUGE THEORY

G. ALDAZABAL, E. DAGOTTO, A. GONZÁLEZ-ARROYO [1] and N. PARGA
Centro Atómico Bariloche [2], Instituto Balseiro [3], 8400-Bariloche, Argentina

Received 21 March 1983

We study the stochastic quantization for lattice QED and derive the corresponding Langevin equations for the Wilson loops. The equivalence with the usual quantization procedure is established in the framework of strong coupling perturbation theory showing that the Wilson loop stochastic equations become the Schwinger–Dyson equations for the loops when the system reaches its equilibrium distribution.

The stochastic quantization [1] of field theories provides a new and powerful tool to deal with these systems.

The basic idea of this procedure is to add a new time dimension t such that the quantum field theory results are attained only when the system reaches its equilibrium distribution at $t \to \infty$. The dependence of the fields on t is determined by a Langevin equation. An interesting feature of the stochastic quantization is that when it is applied to gauge field theories it is not necessary to fix the gauge.

In ref. [1] the stochastic quantization of gauge theories was done by proposing Langevin equations for the potentials. An interesting alternative would be to write down directly the stochastic equations satisfied by the gauge invariant quantities of the theory. Following this approach we report a stochastic quantization of lattice quantum electrodynamics (QED); its equivalence with the standard partition function approach [2] is proved at all orders in the strong coupling expansion by showing that the Langevin equations for the loops become the Schwinger–Dyson equations [3] as $t \to \infty$.

The partition function for QED on a lattice is

$$Z = \int \prod_{\ell} \mathrm{d}\theta_{\ell} \exp\Big(\beta \sum_{\mathrm{P}} \cos\theta_{\mathrm{P}}\Big) , \tag{1}$$

where the ℓ's refer to the links and P to the plaquettes. We propose the following Langevin equation for the independent variables $\theta_{\ell}(t)$

$$\partial\theta_{\ell}/\partial t = -\beta \sum_{\{\mathrm{P}_{\ell}\}} \sin\theta_{\mathrm{P}_{\ell}} + \eta_{\ell}(t) , \tag{2}$$

here $\{\mathrm{P}_{\ell}\}$ is the set of all the plaquettes attached to the link ℓ and $\eta_{\ell}(t)$ is a gaussian stochastic force defined at that link satisfying

$$\langle\eta_{\ell}(t)\rangle_{\eta} = 0 , \tag{3a}$$

$$\langle\eta_{\ell}(t)\,\eta_{\ell'}(t')\rangle_{\eta} = 2\delta(t-t')\,\delta_{\ell\ell'} , \tag{3b}$$

and higher moments are obtained by the Wick decomposition.

From eq. (1) we can find a differential equation for $U_{\ell}(t) = \exp[\mathrm{i}\theta_{\ell}(t)]$

$$\frac{\partial U_{\ell}}{\partial t} = -\tfrac{1}{2}\beta U_{\ell} \sum_{\{\mathrm{P}_{\ell}\}} (U_{\mathrm{P}_{\ell}} - U^{*}_{\mathrm{P}_{\ell}}) + \mathrm{i}\eta_{\ell} U_{\ell} , \tag{4}$$

where U_{P} is the product of the four U_{ℓ}'s of the plaquette P.

From here it is easy to obtain a stochastic equation for an arbitrary Wilson loop $W_{\mathrm{C}}(t) = \Pi_{\ell\in\mathrm{C}} U_{\ell}(t)$. Using eq. (4):

[1] Permanent address: Departamento de Física Teórica C-XI, Universidad Autónoma de Madrid, Cantoblanco, Madrid, Spain.
[2] Comisión Nacional de Energía Atómica.
[3] Comisión Nacional de Energía Atómica and Universidad Nacional de Cuyo.

0 031-9163/83/0000–0000/$ 03.00

$$\frac{\partial W_{\mathrm{c}}(t)}{\partial t}=\frac{\partial}{\partial t}\left(\prod_{\ell\in\mathrm{C}} U_\ell\right)$$

$$=-\tfrac{1}{2}\beta W_{\mathrm{c}}(t)\sum_{\substack{\ell\in\mathrm{C}\\ \{\mathrm{P}_\ell\}}}(U_{\mathrm{P}_\ell}-U^*_{\mathrm{P}_\ell})+\mathrm{i}\eta_{\mathrm{c}}W_{\mathrm{c}}(t)\,, \quad (5)$$

with $\eta_{\mathrm{c}}=\Sigma_{\ell\in\mathrm{C}}\,\eta_\ell$.

These equations can be solved iteratively in a strong coupling (small β) expansion. First we propose

$$W_{\mathrm{c}}(t)=\exp\left(\mathrm{i}\int_0^t \eta_{\mathrm{c}}(t')\,\mathrm{d}t'\right)V_{\mathrm{c}}(t)\,. \quad (6)$$

The first factor is the solution of eq. (5) for $\beta=0$ and the series expansion of $V_{\mathrm{c}}(t)$ is of the form

$$V_{\mathrm{c}}(t)=\sum_{n=0}^{\infty}(\beta/2)^n\,v_{\mathrm{c}}^{(n)}(t)\,, \quad (7)$$

with

$$v_{\mathrm{c}}^{(0)}(t)=1\,. \quad (8)$$

Combining eqs. (5)–(8) we obtain for the nth order of the Wilson loop

$$W_{\mathrm{c}}^{(n)}(t)=-W_{\mathrm{c}}^{(0)}(t)\sum_{k=0}^{n-1}\int_0^t \mathrm{d}t'\,v_{\mathrm{c}}^{(k)}(t')$$

$$\times\sum_{\substack{\ell\in\mathrm{C}\\ \{\mathrm{P}_\ell\}}}[U_{\mathrm{P}_\ell}^{(n-1-k)}(t')-U_{\mathrm{P}_\ell}^{*(n-1-k)}(t')]\,. \quad (9)$$

As an example we calculate the first two orders of $U_{\mathrm{P}}(t)$. The first one is

$$\langle U_{\mathrm{P}}^{(0)}(t)\rangle_\eta=\frac{1}{z}\int \mathcal{D}\eta_\ell(t)\exp\left(\mathrm{i}\int_0^t \eta_{\mathrm{P}}(t')\,\mathrm{d}t'\right), \quad (10)$$

with

$$\mathcal{D}\eta_\ell(t)=\left[\sum_{\ell,t}\mathrm{d}\eta_\ell(t)\right]\exp\left(-\tfrac{1}{4}\int_0^\infty \eta_\ell^2(t')\,\mathrm{d}t'\right), \quad (11\mathrm{a})$$

and

$$z=\int\mathcal{D}\eta_\ell(t)\,. \quad (11\mathrm{b})$$

We see that times in the interval $[t,\infty]$ contribute with a factor one while times in $[0,t]$ give $\exp(-t)$ for each of the links of the plaquette P, then

$$U_{\mathrm{P}}^{(0)}(t)=\exp(-4t)\to 0\,,\quad \text{as } t\to\infty\,. \quad (12)$$

The next order is

$$U_{\mathrm{P}}^{(1)}(t)=-\frac{\beta}{2z}\int\mathcal{D}\eta_\ell(t)\exp\left(\mathrm{i}\int_0^t\eta_{\mathrm{P}}(t')\,\mathrm{d}t'\right)$$

$$\times\int_0^t\mathrm{d}t'\sum_{\substack{\ell\in\mathrm{P}\\ \{\mathrm{P}_\ell\}}}\left[\exp\left(\mathrm{i}\int_0^{t'}\eta_{\mathrm{P}_\ell}(t'')\right)-\mathrm{c.c.}\right]. \quad (13)$$

The reason why eq. (12) is zero as $t\to\infty$ is that in eq. (10) there is a phase contributing at all times from 0 to t. In eq. (13) however it is possible to obtain a non-zero result by cancelling the first phase by those inside the sum over plaquettes. This corresponds to choose only the plaquette P in the set $\{\mathrm{P}_\ell\}$ and the last term of that sum. Since P appears four times we have

$$U_{\mathrm{P}}^{(1)}(t)=\tfrac{1}{2}\beta\,4\int_0^t\mathrm{d}t'\,\exp[-4(t-t')]$$

$$\simeq\tfrac{1}{2}\beta\,,\quad \text{as } t\to\infty\,. \quad (14)$$

A similar reasoning allows to evaluate higher orders in the limit of large t.

In order to relate the Langevin equation (5) with the Schwinger–Dyson equations we first prove that at all orders in perturbation theory

$$\langle W_{\mathrm{c}}(t)\,\eta_\ell(t)\rangle_\eta=\mathrm{i}\langle W_{\mathrm{c}}(t)\rangle_\eta\,, \quad (15)$$

where ℓ belongs to C. This is true because at a given order n a typical term on the left-hand side is of the form

$$z^{-1}(\tfrac{1}{2}\beta)^n\int\mathcal{D}\eta_\ell(t)\exp\left(\mathrm{i}\int_0^t\eta_{\mathrm{c}}(t')\,\mathrm{d}t'\right)\eta_\ell(t)$$

$$\times\int_0^t\mathrm{d}t'\,U_{\mathrm{P}_1}^{(0)}(t')\int_0^{t'}\mathrm{d}t''\,U_{\mathrm{P}_2}^{(0)}(t'')\cdots$$

$$\times\int_0^{t^{(n-1)}}\mathrm{d}t^{(n)}\,U_{\mathrm{P}_n}(t^{(n)})\,, \quad (16)$$

and except for a finite number of points the integral over $\eta_\ell(t)$ is

$$\frac{\int_{-\infty}^{\infty} d\eta_\ell(t) \exp[-\frac{1}{4}\eta_\ell^2(t)\,\Delta t + \frac{1}{2}\mathrm{i}\Delta t\, \eta_\ell(t)]\, \eta_\ell(t)}{\int_{-\infty}^{\infty} d\eta_\ell(t) \exp[-\frac{1}{4}\eta_\ell^2(t)\,\Delta t]} = \mathrm{i}\,, \tag{17}$$

and the relation (15) is verified.

Averaging over $\eta(t)$ in eq. (5) and using eq. (15) we obtain (L is the perimeter of the loop C)

$$\left\langle\frac{\partial W_c(t)}{\partial t}\right\rangle_\eta = -\frac{\beta}{2} \sum_{\substack{\ell\in C\\ \{P_\ell\}}} [\langle W_c(t)(U_{P_\ell} - U^*_{P_\ell})\rangle_\eta] - L\langle W_c(t)\rangle_\eta\,, \tag{18}$$

and since as $t \to \infty$ the left-hand side averages to zero

$$L\langle W_c(t)\rangle_\eta = \frac{\beta}{2} \sum_{\substack{\ell\in C\\ \{P_\ell\}}} [\langle W_c(t)\, U^*_{P_\ell}(t)\rangle_\eta - \langle W_c(t)\, U_{P_\ell}(t)\rangle_\eta]\,. \tag{19}$$

The first (second) term in the sum corresponds to a loop obtained from C by adding on the link ℓ a new plaquette with an opposite (the same) orientation to that of C.

We recognize in eq. (19) the Schwinger–Dyson equation of QED on a lattice [3].

If the loop C is such that a given link ℓ is traversed twice we have to modify eq. (15). In this case we have

$$\langle W_c(t)\,\eta_\ell(t)\rangle_\eta = 2\mathrm{i}\langle W_c(t)\rangle_\eta\,,$$

if the link is traversed in the same direction;

$$\langle W_c(t)\,\eta_\ell(t)\rangle_\eta = 0\,,$$

if it is traversed in opposite directions.

But this is also in agreement with the Schwinger–Dyson equations for this kind of loops.

This completes our proof that the Langevin equation for the Wilson loops, eq. (5), reduces to the Schwinger–Dyson equations giving a correct stochastic quantization of lattice QED. Let us remark that although the proof was proposed for this particular system it can also be applied to scalar or to non-abelian field theories. In the first case it would be an alternative and much simpler proof to the usual diagrammatic one [1,4].

This work was done under a Multinational Program of the Organization of American States.

References

[1] G. Parisi and Wu Yongshi, Sci. Sinica 24 (1981) 483.
[2] K.G. Wilson, Phys. Rev. D14 (1974) 2455.
[3] D. Foerster, Phys. Lett. 87B (1979) 87;
T. Eguchi, Phys. Lett. 87B (1979) 91;
D. Weingarten, Phys. Lett. 87B (1979) 97.
[4] W. Grimus and H. Hüffel, CERN preprint TH-3446 (1982).