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NUMERICAL EXPERIMENTS ON THE SOLUTION OF THE HELMHOLTZ EQUATION IN THE CASE OF DOMAINS OF COMPLICATED BOUNDARY SHAPE

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Domains of complicated boundary shape are of great practical importance in several fields of technology and applied science; e.g. solid propellant rocket grains, electromagnetic and acoustic waveguides, and certain elements used in nuclear engineering. The technical literature contains very few comparative studies of analytical and numerical solutions when dealing with such rather complex geometries. The present study constitutes an effort in that direction.

1. Introduction

The Helmholtz equation is of fundamental importance in several fields of applied science. It describes the behavior of a membrane subjected to transverse vibration, wave propagation in electromagnetic and acoustical waveguides, oscillations of an enclosed body of water, etc. [1].

Recently, the Helmholtz partial differential equation has been used to analyze the "modus operandi" of the human mitral valve [2].

It is important to point out that significant developments in applied mathematics have taken place in the last twenty years due to difficult analytical and computational requirements brought up by electromagnetic applications [3,4].

As stated by Hine [5]: "the study of acoustic wave propagation in ducts is of current research interest in nuclear technology for instance in the case of gas-cooled nuclear reactors where slender fuel rods are hung singly or in clusters in vertical, circular fuel channels. The gas flowing in these channels is typically CO_2 and circulated, at approximately 30 atm, by low-speed gas blowers."

Since the resulting pressure levels are high, it is necessary for reactor designers to estimate the acoustically induced vibration levels and the dynamic stresses generated in the reactor fuel rods [5].

Essentially the acoustic problem reduces to the solution of the Helmholtz equation in the case of domains of "exotic" geometric configurations.

The present paper deals with some numerical experiments on the determination of eigenvalues in problems governed by the Helmholtz equation and in the case of domains of complicated boundary shape.

Simply and doubly connected regions are investigated. Frequency coefficients are calculated in the present study using a finite element code. The results are compared with values obtained by means of:

(a) a conformal mapping-variational method;
(b) a direct variational approach;
(c) the finite differences technique, or
(d) an exact solution in the case of an equilateral membrane. *

* The problem of a vibrating membrane with a fixed boundary is governed by the same differential system which describes wave propagation in an electromagnetic waveguide in the case of TM modes or an acoustic wave which propagates in a "soft" acoustical waveguide.

2. Finite element solution

Since the method is available in standard references [6] no details will be given here. Triangular elements and linear interpolation of the unknown variables are used.*

Fig. 1 shows two membranes of complicated boundary shape and the corresponding finite element distribution (153 nodes and 256 elements for the sector defined in the z-plane $0 \leqslant \Psi \leqslant \pi/n$).

Defining the real plane by z, where

$$z = x + \mathrm{i}y \, , \tag{1}$$

shapes such as those shown in figs. 1(a) and 1(b) are mappable onto a unit circle in the ξ-plane by means of the holomorphic function [10,11]: **

$$z = \frac{a}{1+m} \xi(1 + m\xi^n) \, , \tag{2}$$

where a is the radius of the circle which circumscribes the given shape, n is the number of axes of symmetry, and m is a real number such that $m \leqslant 1/(n+1)$.

Fig. 2 depicts a finite element distribution for a sector of an octagonal membrane (153 nodes and 256 elements are used). It is well known that normal modes of vibration of thin, elastic plates of rectilinear sides and simply supported along the boundary are also governed by the Helmholtz equation [7].

Fig. 3 shows a triangular, equilateral plate which has been studied in refs. [8] and [9].

It is important to point out that knowledge of the dynamic behavior of plates of regular polygonal shape is important in the design of printed circuit boards when they are used in environments subjected to severe vibration requirements. Printed circuit boards of complicated boundary shape are commonly used nowadays.

The case of a membrane of doubly connected domain is illustrated in fig. 4(a) and the corresponding finite element distribution for a sector of the domain is shown in fig. 4(b).

* The computer program has been implemented in an IBM 360/44 computer (Centro Atómico Bariloche, CNEA, Argentina).

** A complex variable-variational approach is used in these two studies to determine fundamental eigenvalues.

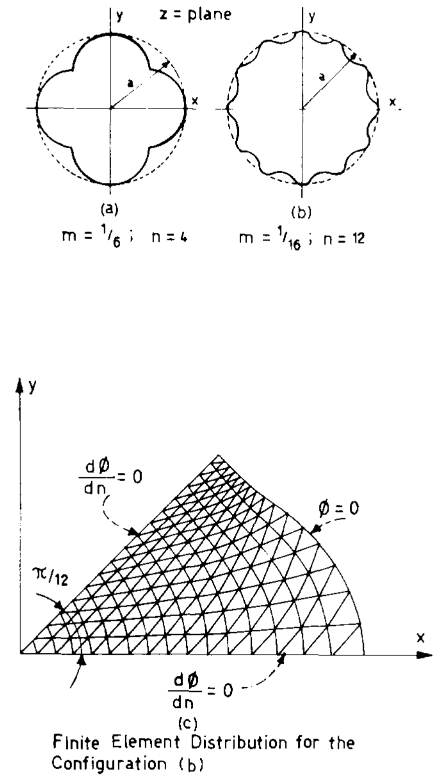

Fig. 1. A class of circular membranes with boundary disturbances.

3. Discussion of results

Table 1 shows a comparison of results of eigenvalues for the configuration shown in fig. 1. The fundamental eigenvalue is denoted by Ω_{01}. The

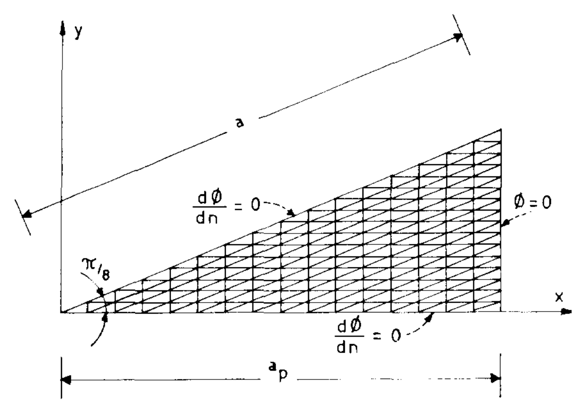

Fig. 2. Sector of an octagonal membrane.

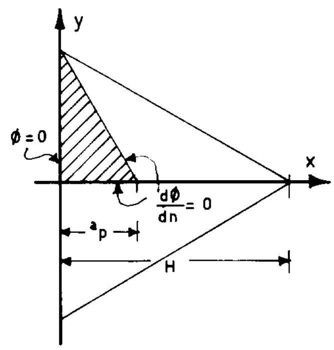

Fig. 3. Triangular, equilateral simply supported plate. *Note*: because of symmetry considerations, it is only necessary to analyse the shaded area (153 nodes and 256 elements are used).

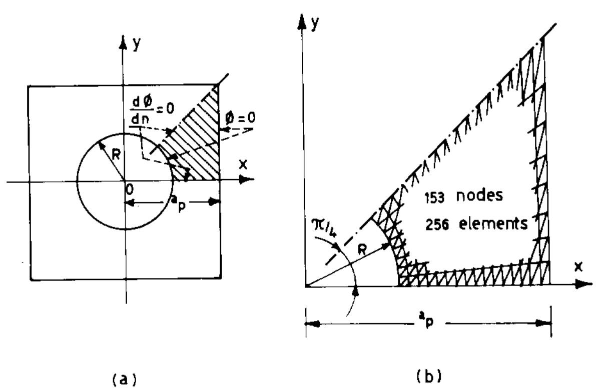

Fig. 4. Square membrane with an inner, circular concentric fixed boundary.

second frequency coefficient, Ω_{02}, corresponds to the normal mode possessing an inner, closed nodal curve [10,11]. From an engineering viewpoint the agreement is very good in al cases.

Table 2
Comparison of frequency coefficients in the case of an octagonal membrane, $\Omega_{0i} = \sqrt{(\rho/S)}\ \omega_{0i}a$

Method	Ω_{01}	Ω_{02}
Conformal mapping – Galerkin method [12]	2.547	–
Conformal mapping – Collocation along arcs [12]	2.549	5.850
Upper bound [13]	2.547	–
Finite elements [14]	2.554	5.986
Finite elements – present study	2.492	5.844

Table 2 deals with the case of the octagonal membrane.

The fundamental eigenvalue obtained in the present study is about 2% lower than the one previously published by other researchers. On the other hand, the second frequency coefficient is in excellent agreement with that obtained in ref. [12].

Fig. 5 compares results for the normalized fundamental mode of vibration of a triangular equilateral plate. The agreement is, in general, very good (only the experimental results obtained by means of a fotonic sensor do not agree with the other experimental, analytical and numerical evaluations).

The fundamental frequency coefficient determined in [9] is

$$\Omega_{01} = \sqrt{(\rho h/D)}\ \omega_{01}H^2 = 39.72\ ,$$

and the one obtained in the present study is

$$\Omega_{01} = 39.52\ .$$

Fig. 6 depicts the variation of the fundamental cut-off frequency of a waveguide with a cross section identical to the shape of the membrane shown in

Table 1
Comparison of frequency coefficients (fig. 1), $\Omega_{0i} = \sqrt{(\rho/S)}\ \omega_{0i}a$

m	*n*	Frequency coefficients	Present study	Ref. [10]	Ref. [11]
1/6	4	Ω_{01}	2.763	2.786	2.792
		Ω_{02}	6.102	6.30	6.36
1/9	7	Ω_{01}	2.638	2.668	2.667
		Ω_{02}	6.060	6.10	6.11
1/16	12	Ω_{01}	2.554	2.555	2.5554
		Ω_{02}	5.879	5.86	5.87

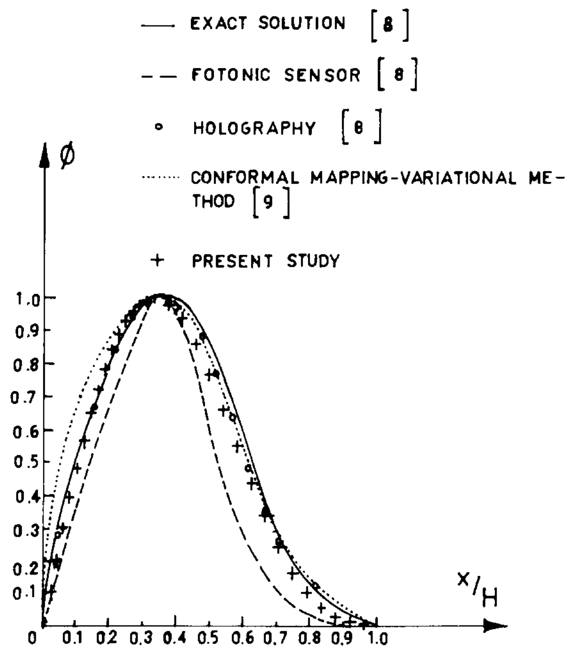

Fig. 5. Normalized fundamental mode of a triangular equilateral plate ($y = 0$).

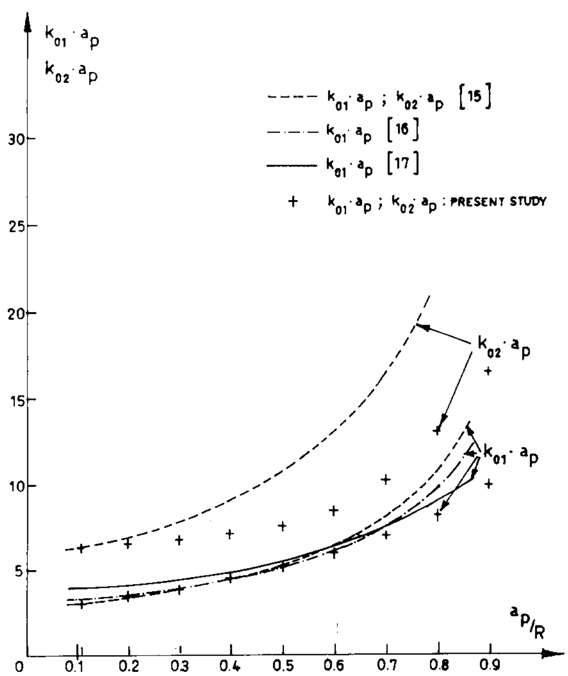

Fig. 6. Comparison of the fundamental cut-off frequency in a square coaxial waveguide (TM-mode).

fig. 4. Obviously when dealing with a membrane one has $k_{0i} = \sqrt{(\rho/S)}\, \omega_{0i}$. Generally the agreement is very good for $0.1 \leqslant a_p/R \leqslant 0.6$ (the results obtained in [17] are higher than those obtained by other techniques). On the other hand, for $a_p/R > 0.6$, the values obtained in [17] are probably the most accurate available in the open literature (since they are upper bounds), and the agreement with the eigenvalues obtained in the present investigation is reasonable from an engineering viewpoint.

It is important to point out that the fundamental cut-off frequency coefficient calculated by Lee and Christian [18] using a finite difference scheme virtually coincides, for the scale chosen to draw fig. 6, with the results obtained in ref. [16].

No claim of originality is made in the present study but it is hoped that it will serve some useful purpose to engineers and applied scientists when dealing eigenvalue problems in domains of complicated boundary shape.

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