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VARIATIONAL DISCUSSION OF THE HAMILTONIAN Z(N) SPIN MODEL IN 1 + 1 AND 2 + 1 DIMENSIONS

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We study the Z(N) spin model, as well as its limiting forms for $N \to \infty$ by means of a variational approach. We find, for 1 + 1 dimensions, the two transitions of the model separating the disordered, massless and ordered phases. In the case of 2 + 1 dimensions, we obtain only the disorder-order phase transition which implies for $N \to \infty$ a single confining phase for the dual U(1) gauge theory.

1. Introduction

The study of the Z(N) spin model has recently received much attention [1] both for its intrinsic interest in phase-transition problems and as a laboratory for the field theories which are supposed to describe high-energy physics.

Here we study the hamiltonian version of this model in 1 + 1 and 2 + 1 dimensions.

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For the case of $1+1$ dimensions the Z(N) model is self-dual and two phase transitions for sufficiently large N are expected from different analyses separating disordered, massless and ordered phases. Two limiting forms of the model emerge naturally for $N \to \infty$ according to the way in which the coupling constant is rescaled: the XY or U(1) spin model which has for this dimensionality disordered and massless phases [2], and the discrete gaussian model which exhibits a massless and an ordered phase [3]. It appears to be rather difficult to obtain a clear picture of both phase transitions with approximate methods or Monte Carlo simulations [4]. In this paper we approach this problem by means of a variational treatment which provides a simple description of the phase structure.

Our procedure consists in analyzing the Z(N) model starting either from the disordered or the ordered phase. The hamiltonian formulation of the theory is based on the operators P and Q with the commutation relation $QP = \exp(-i2\pi/N)PQ$. In the disordered phase we use a mean-field trial state which is built by the same superposition of eigenstates of P at every site. In the ordered phase we start from a state with a definite eigenvalue of $Q = \exp(i2\pi q/N)$ on every site and we allow for random jumps $\Delta q = \pm 1$. Due to the self-duality of the hamiltonian, this state is dual to that used for the disordered phase. An improved version is obtained by considering random jumps among the peaks of a superposition of eigenstates of Q which, again by duality, corresponds to an improvement of the mean-field state. With this technique we find a disorder-massless phase transition for a small critical coupling constant $\sim 1/N^2$ and a massless-order transition for a large critical coupling $\sim N^2$.

For the case of $2+1$ dimensions it is expected that there is just one transition between the disordered and ordered phases in the Z(N) model, and consequently in the XY limiting form. This transition appears for a value of the coupling constant $\sim 1/N^2$ and is obviously obtained with our previously discussed mean-field trial state simply by changing the dimensionality. The presence of a further transition at a coupling constant $\sim N^2$ would imply a transition for the discrete gaussian model and also for the U(1) gauge model in $2+1$ dimensions which is related to it by duality. Numerical studies on the U(1) gauge model [5], field-theoretical considerations [6] and a recent rigorous discussion [7] indicate that this phase transition is indeed absent.

In order to analyze this last point we have first considered a state which is the product of the same superposition of P-eigenvectors on every site. This state obviously presents the above-mentioned low-λ transition, having thereafter a continuous behavior which ends as an eigenvector of Q for $\lambda \to \infty$. To study the possibility of an intermediate phase we have also taken a state based on a domain spin structure with minimal jumps across the walks. It turns out that the energy of this last state coincides for large λ with the previously mentioned one but becomes higher when jumps greater than the minimal one ($\Delta q = \pm 1$) are important. For an intermediate value of λ this domain ansatz develops a non-analyticity corresponding to a domain condensation which is, however, unphysical since its energy is not the minimum in the considered set of states. Moreover, we observe that the Wilson loop for the U(1) gauge model calculated with the trial states appropriate to the strong and weak coupling limit always shows an area behavior. Our conclusion is therefore in agreement with refs. [5–7]. Since the discrete gaussian model is the high-fugacity limiting form of the classical Coulomb gas problem, from this analysis it follows that there is no dielectric-plasma transition in 2 + 1 dimensions.

2. Z(N) and related models

We shall be interested in the phase transitions of the global symmetric Z(N) model in $d+1$ dimensions and of those which may be derived from it.

The hamiltonian for the Z(N) model is

$$H = -\tfrac{1}{2}\sum_s (P_s + P_s^\dagger) - \lambda\tfrac{1}{2}\sum_{\langle ss'\rangle} (Q_s Q_{s'}^\dagger + Q_s^\dagger Q_{s'}), \tag{1}$$

where, e.g., in the Q representation

$$Q_{nn'} = e^{i2\pi n/N}\delta_{nn'}, \qquad P_{nn'} = \delta_{n,n'-1}, \tag{2}$$

with $n = 0, 1, \ldots, N-1$ modulo N.

Introducing $Q = \exp(i2\pi q/N)$ for $N \to \infty$ and large λ, the minimum of H will come from $\cos[2\pi(q_s - q_{s'})/N] \sim 1$, so that H will take the form, apart from a constant,

$$H_{\mathrm{DG}} = -\tfrac{1}{2}\sum_s (P_s + P_s^\dagger) + \omega\sum_{\langle ss'\rangle} (q_s - q_{s'})^2, \qquad \omega = 2\lambda\left(\frac{\pi}{N}\right)^2, \tag{3}$$

where the eigenvalues of q are all the integer numbers. Eq. (3) corresponds to the non-compact discrete gaussian model or Z-ferromagnet, which, for $d = 1$, is used as a realization [8] of the roughening model.

Alternatively, defining

$$Q = \exp(i\theta), \qquad P = \exp(i2\pi L/N),$$

for $N \to \infty$ and small λ, the minimum of H, eq. (1), corresponds to $\cos(2\pi L/N) \sim 1$. In this limit θ is a continuous 0–2π angle and L an angular momentum operator, and the expansion of eq. (1) gives, apart from a subtraction and a rescaling factor,

$$H_{XY} = \sum_s \tfrac{1}{2}L_s^2 - \bar{\omega}\sum_{\langle ss'\rangle} \cos(\theta_s - \theta_{s'}), \qquad \bar{\omega} = \lambda\left(\frac{\pi}{2N}\right)^2. \tag{4}$$

The hamiltonian of eq. (4) corresponds to the XY or U(1) global model.

Fig. 1. Direct sites (dots) involved in the calculation of (a) $\tilde{P}$ and (b) $\tilde{Q}$ at one dual site (cross) for the $1+1$ Z(N) spin model.

Finally, if we consider the sine-Gordon model on the lattice,

$$H_{\mathrm{SG}} = \sum_s \tfrac{1}{2} p_s^2 + \tfrac{1}{2} r \sum_{\langle ss' \rangle} (x_s - x_{s'})^2 + h \sum_s \sin^2 x_s , \tag{5}$$

where x_s and p_s are canonical variables for each site, the limit of large h forces x to be $\sim n\pi$ and p^2 to have, because of tunnelling, only matrix elements between at most neighbouring localized states. Therefore in this limit H_{SG} is equivalent to H_{DG} of eq. (3).

If we examine the case of $d=1$, we find that the Z(N) hamiltonian model of eq. (1) is self-dual with (see fig. 1)

$$\tilde{Q}_s = \prod_{s'<s} P_{s'}^{\dagger}, \qquad \tilde{P}_s = Q_s Q_{s+1}^{\dagger}, \qquad \tilde{\lambda} = 1/\lambda . \tag{6}$$

It has been obtained by several methods, and will be described in the next section by means of the variational approach, that for sufficiently large N the model has three phases: a disordered phase for $\lambda < \lambda_1 \propto 1/N^2$, an ordered one for $\lambda > \lambda_2 = 1/\lambda_1$ and an intermediate massless phase for $\lambda_1 < \lambda < \lambda_2$. As a consequence, the discrete gaussian model of eq. (3) will have a single transition between the massless and the ordered phases. The same transition will appear for the sine-Gordon model eq. (5) for large h. Analogously, the XY model of eq. (4) will have the well-known Kosterlitz-Thouless transition between the disordered and the massless phases.

Going now to the case of $2+1$ dimensions, the hamiltonian, eq. (1), is not self-dual but dual to the Z(N) gauge model according to the transformation [9] (see fig. 2)

$$P_s = (\tilde{Q}\tilde{Q}\tilde{Q}^{\dagger}\tilde{Q}^{\dagger})_p , \qquad Q_s = \prod_{<} \tilde{P}_l^{\dagger} , \qquad \lambda = 1/\tilde{\lambda} , \tag{7}$$

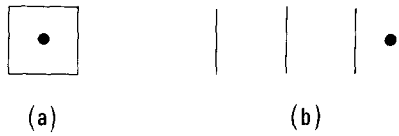

Fig. 2. Links of $2+1$ Z(N) gauge model involved in the calculation of (a) P and (b) Q Z(N) spin variables at one site (dot).

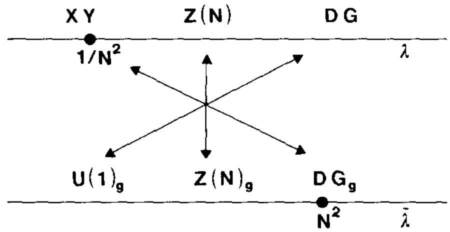

Fig. 3. Schematic relation between the Z(N) model in 2 + 1 dimensions and its limits XY and discrete gaussian (DG) model, with the gauge model Z(N)$_g$, and its limiting gauge models U(1)$_g$ and DG$_g$.

where gauge variables are defined on links of the dual lattice. The dual gauge theory has as an $N \to \infty$ limit for small $\tilde{\lambda} = 1/\lambda$ the U(1) gauge model and for large $\tilde{\lambda}$ the gauge discrete gaussian model [10] (see fig. 3).

It is now clear that if, as seems likely, the spin Z(N) model has only one phase transition at $\lambda \sim 1/N^2$ between disordered and ordered phases the global discrete gaussian model has only one ordered massive phase that in the continuous limit, which one recovers around the critical point $\omega = 0$ (∞ temp), corresponds to the massless phase characteristic of the free field. The XY model has also a single order-disorder transition at a finite value of $\bar{\omega}$ and the sine-Gordon model shows only one ordered phase. Conversely, for the gauge case the situation is reversed by duality: the U(1) gauge theory will exhibit a single confining phase, whereas the gauge discrete gaussian model will have two phases.

The presence of a second transition for the spin Z(N) model at $\lambda \sim N^2$ giving rise to a massless phase would imply a transition in the discrete gaussian model and a deconfining transition in U(1) gauge theory contrary to the common belief.

3. Variational approach in 1 + 1 dimensions

It is easy to obtain an estimate of the two phase transitions of the Z(N) model, eq. (1), for $N \gtrsim 5$ using a variational method already proposed in ref. [11].

Starting from very low values of λ, the ground state is reasonably described by

$$|\varepsilon\rangle = \prod_{\text{sites}} \frac{1}{\sqrt{1+2\varepsilon^2}}\left[|0\rangle_P + \varepsilon(|1\rangle_P + |-1\rangle_P)\right], \tag{8}$$

where $P|n\rangle_P = \exp(i2\pi n/N)|n\rangle_P$. This state is of the "mean-field" type since all sites are equal. The expectation value per site of H, eq. (1),

$$E(\varepsilon) = -\frac{1+2\varepsilon^2\cos(2\pi/N)}{1+2\varepsilon^2} - \lambda\left(\frac{2\varepsilon}{1+2\varepsilon^2}\right)^2,$$

is minimized by $\varepsilon = 0$ for $\lambda < \lambda_1 = \frac{1}{2}(1 - \cos(2\pi/N))$, whereas ε becomes > 0 with continuity for larger couplings. The fact that the transition occurs at a small value of λ (for large N) justifies the use of the state equation (8) where only the eigenstates energetically closest to $|0\rangle_P$ are included. As a check, it has been seen that considering a superposition of all the eigenvectors $|n\rangle_P$ the change in the numerical value of λ_1 is irrelevant. The critical coupling for the XY model is predicted to be $\bar{\omega} = 0.25$ compared to the value ~ 0.8 obtained by perturbative computations [12].

The shortcoming of this treatment is that the variational state, eq. (8), produces for $\lambda > \lambda_1$ a non-vanishing order parameter $\langle Q_s \rangle$ and a jump from 1 to 0 of the disorder parameter $\langle \tilde{Q}_s \rangle$ [see eq. (6)], whereas the correct feature would be the continuous vanishing of the latter keeping the former equal to zero. This depends on having chosen a strictly "mean-field" state, which is very rough since, e.g. for $\lambda < \lambda_1$, it does not include the perturbative corrections in λ. The situation may be improved by using more complex trial states, and better described in terms of dual variables, as will be explained below, which turn out to be equivalent to taking into account these corrections in λ.

To study the second phase transition of Z(N) in 1 + 1 dimensions for large λ, the obvious trial state built in terms of eigenstates of Q,

$$|\mu\rangle = \prod_{\text{sites}} |l, \mu\rangle = \prod \left[|l\rangle_Q + \mu\left(|l+1\rangle_Q + |l-1\rangle_Q \right) \right] \Big/ \sqrt{1 + 2\mu^2}\,, \tag{9}$$

where $Q|l\rangle = \exp(i2\pi l/N)|l\rangle$, gives an analytic relation $\mu(\lambda)$. On the contrary, the "random-walk" state

$$|\xi\rangle = \sum_{\{l\}} \prod_{\text{sites}} C(l_s - l_{s-1})|l_s\rangle_Q\,, \tag{10}$$

with non-vanishing coefficients $C(0) = 1$, $C(\pm 1) = \xi$, corresponds to $\xi = 0$ for $\lambda > \lambda_2 = 2/(1 - \cos(2\pi/N))$, becoming $\xi > 0$ with continuity for lower values of λ.

The two critical points λ_1 and λ_2, though possibly too low and too large, respectively, because of the lack of fluctuations, are clearly distinct for large N. It is, moreover, easy to verify that $E(\varepsilon) > E(\xi)$ for $\lambda \simeq \lambda_2$ and vice versa for $\lambda \simeq \lambda_1$ showing that each variational state is acceptable in its proper region. The problems are again that the state $|\xi\rangle$ produces a discontinuous change of $\langle Q \rangle$ from 1 to zero and that $\langle \tilde{Q} \rangle$ becomes > 0 for $\lambda < \lambda_2$. This is also evident from the fact that the state of eq. (10) is the dual of the state of eq. (8), i.e. $|\xi\rangle_Q = |\varepsilon\rangle_{\tilde{P}}$, being, therefore, a mean-field state in terms of dual variables.

These difficulties are partially solved if one builds a "random-walk" state in terms not of eigenstates of Q as in eq. (10), but of the broader states $|l, \mu\rangle_s$, eq. (9), i.e.

$$|\xi, \mu\rangle = \sum_{\{l\}} \prod_{\text{sites}} C(l_s - l_{s-1})|l_s, \mu\rangle\,. \tag{11}$$

We anticipate the result that, in so doing, the energy will be minimized for $\xi = 0$, $\mu \neq 0$ when $\lambda > \lambda_2$, giving, therefore, a not completely ordered phase. For $\lambda < \lambda_2$ ξ becomes > 0 producing a lower, though not zero, discontinuity of $\langle Q \rangle$.

When the random walk parameter is turned off we are left with the sharp state $\Pi_s |l\rangle$ in the case of eq. (10) and the broad one of $\Pi |l, \mu\rangle$ with the choice of eq. (11). It is interesting to remark that the latter is equivalent for small μ to the perturbative expansion in $1/\lambda$:

$$|\phi\rangle^{(1)}_{\text{pert}} = \prod_s |l\rangle_s + \frac{1}{\lambda} \frac{1}{2\cos(2\pi/N)} \sum_{\bar{s}} \prod_{s \neq \bar{s}} |l\rangle_s \tfrac{1}{2}(|l+1\rangle_{\bar{s}} + |l-1\rangle_{\bar{s}}), \tag{12}$$

since eq. (12) coincides with the first-order expansion in μ of the state $\Pi_s |l, \mu\rangle_s$. It is obvious that, because of the self-duality of the hamiltonian, eq. (1), a random walk state in terms of dual variables of broad states $|l, \mu\rangle_{\tilde{Q}}$ gives for $\lambda < \lambda_1$ a perturbative correction of eq. (8).

The picture is summarized in fig. 4 where it is seen that a mean-field state corresponding to the same superposition of P-eigenvectors at every site describes correctly the transition at λ_1 and tends smoothly to a Q-eigenvector for $\lambda \to \infty$ without any further non-analyticity. On the other hand, the random walk state gives for $\lambda < \lambda_2$ an energy lower than that of the mean-field state producing therefore the second phase transition. In the region around λ_1 the energy of the random walk state is larger than that of the mean-field ansatz because there jumps larger than $\Delta q = \pm 1$, which we have not taken into account, become relevant.

We come now to the technical description of the variational energy with the random walk of broad states eq. (11). The reader not interested in the details could skip this part. It is convenient to write the total state as a sum of random walks

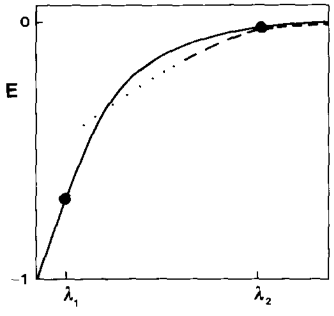

Fig. 4. Energy per unit volume $E = \langle H \rangle / V + \lambda$ as a function of λ in (1 + 1) dimensions for the mean-field state, solid line, and for the random walk state, eq. (13), broken and dotted line. The dotted part is in the region where the ansatz becomes inappropriate.

constrained to pass through the value $\bar{l}$ at site $\bar{s}$:

$$|\xi, \mu\rangle = \sum_{\bar{l}} |\bar{l}, \bar{s}\rangle = \sum_{\bar{l}} \sum_{\{l\}} \prod_{s} \delta_{l_{\bar{s}}, \bar{l}} C(l_s - l_{s-1}) |l_s, \mu\rangle, \tag{13}$$

where the dependence on ξ and μ of $|\bar{l}, \bar{s}\rangle$ is implicit. The norm will be

$$\langle \xi, \mu | \xi, \mu \rangle = \sum_{|\bar{l}-\bar{l}'| \leq 2} \langle \bar{l}' \bar{s} | \bar{l} \bar{s} \rangle = w_0 + w_1 + w_2 = 1. \tag{14}$$

For the calculation of w_j it is useful, for normalizability reasons, to define

$$v_j(0, m) = \left[\prod_{0 \leq s \leq m} \sum_{\{l_s\}} C(l_s - l_{s-1}) \delta_{l_{s=0}, l+j} \langle l_s, \mu | \right] \times \left[\prod_{0 \leq s \leq m} \sum_{\{l_s\}} C(l_s - l_{s-1}) \delta_{l_{s=0}, l} | l_s, \mu \rangle \right]. \tag{15}$$

The relation with the coefficients w_j is, for large m, $w_j \propto v_j^2 / u_j$, where $u_j = \langle l \pm j, \mu | l, \mu \rangle$.

It is an obvious identity to write

$$v_j(0, m) = u_j \sum_{k=0}^{2} \gamma_k u_{j \pm k}(1, m), \tag{16}$$

where γ_k supplies the necessary coefficients to change by k the separation of the two paths at the site $s = 1$, i.e. $\gamma_0 = 1 + 2\xi^2$, $\gamma_1 = 2\xi$, $\gamma_2 = \xi^2$. For sufficiently large m, one expects that the effect of the increase of one site in the calculation of v_j will be independent of j due to the loss of memory of the random walk in the internal compact space about the initial position, i.e.

$$v_j(0, m) = t v_j(1, m). \tag{17}$$

In this way eq. (16) gives a homogeneous system in the three unknowns $v_j = v_{-j}$, and the vanishing of the associated determinant fixes the value of t. Being interested in the region close to the transition point, we keep only first-order terms in ξ obtaining $t = (1 - 4\xi) u_j$. We accept the solution with $j = 0$ by continuation of the expected behavior at $\xi = 0$, i.e. $t = 1$. Therefore, for small μ

$$v_0 = 1, \qquad v_1 = 4\xi\mu, \qquad v_2 = 0. \tag{18}$$

We thus finally obtain the normalized coefficients

$$w_0 = 1 - 8\xi^2\mu, \qquad w_1 = 8\xi^2\mu, \qquad w_2 = 0. \tag{19}$$

Always for small ξ and μ the two pieces of the hamiltonian eq. (1) are

$$\tfrac{1}{2}\langle P + P^{\dagger}\rangle = w_0 \frac{u_1}{u_0} + w_1 \frac{u_0 + u_2}{u_1} = 2\mu + 4\xi^2,$$

$$\tfrac{1}{2}\langle Q_s Q_{s+1}^{\dagger} + Q_s^{\dagger} Q_{s+1}\rangle = v_0^2 \frac{\langle 0\mu|Q|0\mu\rangle}{1+2\xi^2} \sum_{k=-1}^{1} \langle k\mu|Q^{\dagger}|k\mu\rangle\left[\delta_{k0} + (1-\delta_{k0})\xi^2\right]$$

$$= \left(\frac{1+2\mu^2\cos(2\pi/N)}{1+2\mu^2}\right)^2 \left[1 - 2\left(1-\cos\frac{2\pi}{N}\right)\xi^2\right]. \tag{20}$$

Therefore one sees that ξ becomes > 0 for $\lambda < \lambda_2 \simeq N^2/\pi^2$. For $\lambda > \lambda_2$, $\xi = 0$ but the minimum of the variational energy occurs for a value of μ which ranges from $\mu = 0$ for $\lambda \to \infty$ to $\mu \simeq \frac{1}{8}$ for $\lambda = \lambda_2$.

4. The case of 2 + 1 dimensions

We wish to study the Z(N) model in 2 + 1 dimensions with the same techniques as those used for the 1 + 1 case.

If we start from the low-λ region and use the trial state eq. (8), it is obvious that we find the same phase transition, as in any dimension, for $\lambda_1 = (1 - \cos(2\pi/N))/(2d)$. As explained for the 1 + 1 dimensional case, the same superposition of P-eigenvectors in every site (e.g. a gaussian one) shows a transition at this value of λ beyond which no further non-analyticity appears up to $\lambda \to \infty$ where an eigenvector of Q at every site is smoothly reached. The problem is now whether a second phase transition may be induced by a domain state which is a generalization of the random walk state of the 1 + 1 dimensional case.

We give below a brief description of such a generalization. It will turn out that in the present case this ansatz is not favoured by the hamiltonian. A domain is a connected region of the lattice with the same eigenstate of Q at every site, where due to energy considerations we keep only jumps $\Delta q = \pm 1$. The operator for the creation of a domain is, taking into account that the amplitude is expected to decrease with the exponential of the perimeter L,

$$\Omega_{\mathrm{D}} = \varepsilon^L \prod_{s\in \mathrm{D}} P_s, \tag{21}$$

which has to be applied to the perturbative ground state $|0\rangle$ where all the sites have the same eigenvalue of q which we choose to be zero. In eq. (21) ε is a small parameter. Thus we have that the trial ground state is

$$|\phi\rangle = \sum_{\langle \mathrm{D}\rangle} \Omega_{\mathrm{D}}|0\rangle. \tag{22}$$

According to fig. 5a a domain is visualized by a polygon on the dual lattice. The action of the first part of the hamiltonian, eq. (1), i.e. the operator P, displaces the contours of the domain (see fig. 5b), whereas the second term, i.e. the operator $1 - \frac{1}{2}[Q_s Q^\dagger_{s+l} + \text{h.c.}]$ counts the contours passing through the link l. We consider an approximation where the domain excitations are dilute. This means that we may do the computations for just one domain since the others factorize. Therefore we get

$$\tfrac{1}{2}\langle Q_s Q^\dagger_{s'} + \text{h.c.}\rangle = \frac{1 + 2\Sigma_L \varepsilon^{2L} n_L \cos(2\pi/N)}{1 + 2\Sigma_L \varepsilon^{2L} n_L}, \tag{23}$$

where n_L is the number of self-avoiding polygons of length L, and the factor 2 takes into account the positive and negative domains. The number of closed self-avoiding random walks with one fixed link is known numerically [13] for the first values of L and asymptotically from the polymer theory [14]

$$n_L \simeq kc^L / L^{\nu d}, \tag{24}$$

with $d = 2$ and $\nu = \frac{3}{4}$ for our case and [15] $c = 2.6385$ and $k \simeq 0.6$.

The computation of $\langle P_s \rangle$ is more involved because the action of P changes, in general, the length of the domain. We have foreseen the four possibilities shown in fig. 6. In this way we obtain

$$\tfrac{1}{2}\langle P_s + P_s^\dagger \rangle = \frac{2\varepsilon^4 + 4\Sigma_L[\varepsilon^{2L+2} n_L(1) + \varepsilon^{2L} n_L(2) + \varepsilon^{2L-2} n_L(3)]}{1 + 2[\varepsilon^8 + 4\Sigma_{i=1}^{3} \Sigma_L \varepsilon^{2L} n_L(i)]}, \tag{25}$$

where $n_L(i)$ are the numbers of domains of type i as indicated in fig. 6(i) and the factor 4 takes into account the possible space orientations. The terms ε^4 and ε^8 correspond to the type $i = 4$. The multiplicity numbers $n_L(i)$ have been explicitly computed upto $L = 12$ and for larger L we have used again the polymer theory estimations.

For large values of λ the variational energy is minimized by $\varepsilon^2 c < 1$, c appearing in eq. (24), which implies that only small domains are important. At $\lambda = \bar{\lambda} = 0.06/(1 - \cos(2\pi/N))$ the minimum of the energy appears at $\varepsilon^2 c = 1$ which gives a divergent

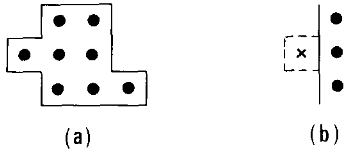

Fig. 5. (a) Domain of equal values of Q on the sites denoted by dots. (b) Deformation of the boundary of a domain due to the action of P on the site denoted by a cross.

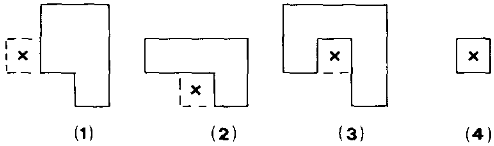

Fig. 6. Deformation of different contours due to the action of P on the site denoted by a cross.

domain perimeter expectation $\langle L\rangle$, corresponding to a domain condensation. However, as shown in fig. 7, this non-analyticity lies in an unphysical region since the mean-field state

$$|\gamma\rangle=\prod_s\sum_n e^{-\gamma n^2}|n\rangle_P \tag{26}$$

produces a lower energy. For instance, for large N the minimum energy with the state of eq. (26) is $\langle H\rangle/V=-0.74$, V being the volume equal to the total number of sites, at $\gamma(\bar\lambda)=0.72\ (2\pi/N)^2$, whereas the domain state gives $\langle H\rangle/V=-0.28$. At such low values of $\lambda\sim\bar\lambda$ the variance of Q with the state of eq. (26) is larger than 1 and the ansatz of the domain state is no longer appropriate since it would require a value for this variance $\ll 1$.

Our final considerations concern the Wilson loop for the Z(N) gauge model which may be evaluated in the Z(N) spin variables according to

$$\left\langle\prod_{l\in C}\tilde{Q}_l\right\rangle=\left\langle\prod_{s\in A}P_s\right\rangle, \tag{27}$$

A being the surface enclosed by the contour C. The mean-field state, i.e. the same superposition either of P or Q eigenvectors at every site, gives the area behaviour.

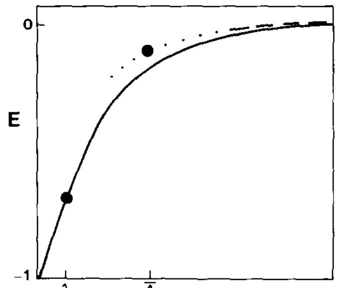

Fig. 7. Energy per unit volume $E=\langle H\rangle/V+2\lambda$ as a function of λ in (2 + 1) dimensions for the mean-field state, solid line, and for the domain state, broken and dotted line.

With the state $|\varepsilon\rangle$, eq. (8), valid for $\lambda \gtrsim \lambda_1$, one has

$$\left\langle \prod_{s\in A} P_s \right\rangle_\varepsilon = \left[\left(1+2\varepsilon^2\cos(2\pi/N)\right)/(1+2\varepsilon^2) \right]^A . \tag{28}$$

For large values of λ the mean-field state smoothly goes into a state $|\mu\rangle$, eq. (9), which gives

$$\left\langle \prod_{s\in A} P_s \right\rangle_\mu = \left[2\mu/(1+2\mu^2) \right]^A . \tag{29}$$

Since this explored region of λ is that which corresponds to all the coupling of the limiting U(1) gauge theory, it follows that the latter always shows an area behaviour consistent with the absence of a phase transition.

Note that the appearance of the domain state, eq. (22), would give a perimeter law since the Wilson loop is the creation operator of a domain. Since $\mu(\lambda) \simeq 1/8(1-\cos(2\pi/N))\lambda$, eq. (29) gives a string tension $\sigma = \ln 4\omega$ for $N \to \infty$ which is the expected results from a strong coupling (large ω) expansion of the U(1) gauge theory. On the other hand, the weak coupling limit (small ω) is obtained using the state $|\gamma\rangle$, eq. (26), in eq. (28) for $\gamma = -\ln \varepsilon$, giving $G = \sqrt{\frac{1}{2}\omega}$. This picture should be compared with Z(2) gauge model in 3 + 1 dimensions where the variational state in dual variables was shown [16] to predict a change from area to perimeter behaviour going from the strong to weak coupling regime.

Our conclusion is therefore that we do not find evidence for a phase transition in the discrete gaussian model, or equivalently for the second large-λ phase transition of the $Z(N)$ model in 2 + 1 dimensions.

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Note added in proof:

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