

## THE NUCLEAR FIELD TREATMENT OF SOME EXACTLY SOLUBLE MODELS

D. R. BÈS<sup>†</sup>

*Comisión Nacional de Energía Atómica, Buenos Aires, Argentina*<sup>††</sup>

R. A. BROGLIA

*Niels Bohr Institute, University of Copenhagen, Denmark*

and

G. G. DUSSEL<sup>†</sup>, R. J. LIOTTA and H. M. SOFÍA

*Comisión Nacional de Energía Atómica, Buenos Aires, Argentina*<sup>††</sup>

Received 17 April 1975

**Abstract:** The nuclear field theory is applied to physically meaningful models which are exactly soluble. The particles are coupled through monopole particle-hole and/or pairing forces, which do not mix states with a different number of particles and holes. It is possible to sum up all the field diagrams and, thus, one obtains the exact results in all cases. Therefore, the overcompleteness of the basis and the violations of the Pauli principle are corrected for within the field treatment.

### 1. Introduction

A field treatment of a residual two-body interaction should have built-in mechanisms in order to correct for the violations of the Pauli principle and for the overcompleteness of the basis, which are inherent to the simultaneous use of both collective and single-particle degrees of freedom. More generally, it should reproduce the results of a conventional shell-model diagonalization of the residual two-body interaction. The feasibility of such field theory was shown in ref. <sup>1</sup>), where the case of particles moving in two levels and coupled through a monopole particle-hole force was treated. In ref. <sup>2</sup>) the overcompleteness of the field basis was increased so as to include particle-particle (pairing) fields. In the present paper we clarify and extend the calculations which were performed in refs. <sup>1,2</sup>) We treat physically meaningful models which are exactly soluble, both for the fermion and the field calculations.

In the field treatment one uses both particle and vibrational degrees of freedom. The energies of the uncoupled particle fields must contain the Hartree-Fock contributions of the two-body residual interaction. The energy of the phonon fields are obtained through a RPA treatment of the same interaction.

Both particle-vibration coupling vertices (fig 1c) and four-point fermion vertices (fig 1b) have to be included. The former represent the matrix elements of the Ham-

<sup>†</sup> Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

<sup>††</sup> Mailing address. Departamento de Física Nuclear, CNEA, Avda del Libertador 8250, Buenos Aires (S 29), Argentina

iltonian linking the normal (RPA) modes with the particle-hole states. These two couplings are allowed to act in all orders to generate the different diagrams of perturbation theory. The contribution of each diagram is evaluated by the usual rules, with two restrictions

(i) Initial and final states are restricted to proper states. Such a state involves collective modes and particle modes, but not any particle configuration that can be replaced by a combination of collective modes. This restriction does not apply to the internal lines of the diagrams.

(ii) The internal lines of diagrams are restricted by the exclusion of diagrams in which a particle-hole pair is created and subsequently annihilated, without having participated in other interactions in between (bubbles).

These rules were stated in ref. <sup>1)</sup> Similar rules appear for the calculation of transition matrix elements corresponding to an operator  $Q$  which creates or destroys a particle-hole pair. In the field treatment one must retain the vertices corresponding to the fermion operator  $Q$  (fig. 3a) and also include the vertices representing the creation and annihilation of a phonon (fig. 3b). The values of these last vertices are given by the matrix element of the operator  $Q$  between the vacuum state and the corresponding RPA normal mode. Similar restrictions as for the energy have to be imposed for the evaluation of the diagrammatic expansion.

The nuclear field theory may also be applied when the collective states correspond to pairing modes. In this case we disregard diagrams including particle-particle bubbles. An operator  $P^+$  which creates two particles has two vertices within the field treatment: one representing the creation of the two fermions, the second one corresponding to the creation or annihilation of a pairing phonon.

## 2. Degenerate particle-hole excitations and a particle-hole monopole interaction

The fermion Hamiltonian  $H$  includes a single-particle term  $H_{s,p}$ , and a two-body interaction  $H_{t,b}$ ,

$$H = H_{s,p} + H_{t,b}, \quad (1)$$

$$H_{s,p} = \frac{1}{2}\varepsilon_0(N_1 + N_{\bar{1}}), \quad (2)$$

$$H_{t,b} = -\frac{1}{2}V(A^+A + AA^+), \quad (3)$$

where

$$A^+ = \sum_m c_{m1}^+ c_{m\bar{1}}; \quad N_1 = \sum_m c_{m1}^+ c_{m1}; \quad N_{\bar{1}} = \sum_m c_{m\bar{1}} c_{m\bar{1}}^+. \quad (4)$$

The operator  $c_{m\sigma}^+$  creates a particle in the state  $(m, \sigma)$ . The quantum number  $\sigma$  ( $\sigma = 1$  or  $\sigma = -1 = \bar{1}$ ) denotes either the upper or the lower level. Each level has degeneracy  $2\Omega$  and the subscript  $m$  labels each degenerate state. Here,  $\varepsilon_0$  is the distance between the two levels and  $V$ , the strength of the monopole particle-hole interaction.

The fermion treatment of the Hamiltonian (1) is carried out in appendix A. The procedure is straightforward, because of the applicability of the quasi-spin formalism<sup>3</sup>).

The fermion terms in the field treatment must include the Hartree-Fock contributions of  $H_{t_b}$ . Since the matrix element of the interaction (3) between the closed shell state  $|\rangle$  and any particle-hole state vanishes, the chosen single-particle basis satisfies the Hartree-Fock minimization condition. Consequently, no change in the single-particle wave function is required. However, there is a contribution to the independent-particle energy  $\varepsilon_\sigma$ , which is defined through the linearization equations

$$[H, c_{m\sigma}^+]|\rangle = \varepsilon_\sigma c_{m\sigma}^+|\rangle \quad (5)$$

We obtain

$$\varepsilon_1 = -\varepsilon_{\bar{1}} = \frac{1}{2}(\varepsilon_0 + V), \quad (6)$$

and write the pure fermion terms in the field treatment as

$$\begin{aligned} H'_{s_p} &= \varepsilon_1 N_1 - \varepsilon_{\bar{1}} N_{\bar{1}} = \frac{1}{2}\varepsilon(N_1 + N_{\bar{1}}), \\ H'_{t_b} &= -VA^+A, \end{aligned} \quad (7)$$

where the Hartree-Fock particle-hole excitation energy is given by

$$\varepsilon = \varepsilon_0 + V \quad (8)$$

The only vertex corresponding to the interaction  $H'_{t_b}$ , is given in fig 1b. The difference between  $H'_{t_b}$  and  $H_{t_b}$  eliminates in this case the Fock single-particle insertions of fig. 1a, which are present for the interaction (3). An amount  $V\Omega$  is added to all energies.

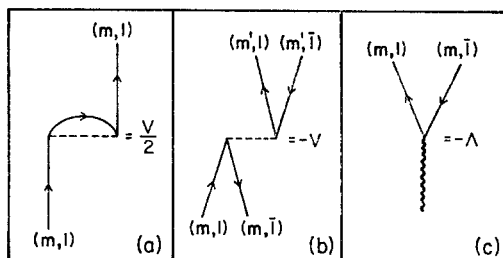


Fig 1 Graph (a) represents the Fock self-energy insertions. Graphs (b) and (c) correspond to the four-point vertex of the interaction (7) and to the vertex of the particle-vibration interaction (11), respectively. The value of the vertices is as indicated in the figure, when the particle line appears to the left of the hole line, for all pairs  $(m, 1, m, \bar{1})$  meeting the vertex. A minus sign appears for each interchange of this order.

The pure boson term  $H_b$  corresponds to a set of independent phonons

$$H_b = \sum_{i=1}^{2\Omega} \omega_i \Gamma_i^+ \Gamma_i, \quad (9)$$

with eigenfrequencies  $\omega_i$  given by the TDA<sup>†</sup>,

$$\omega_i = \begin{cases} \varepsilon - 2V\Omega & (i = 1) \\ \varepsilon & (i = 2, 3, \dots, 2\Omega). \end{cases} \quad (10)$$

The particle-vibration interaction  $H_{p,v}$  may be obtained<sup>4)</sup> replacing first the operator  $A^+$  by  $\sum_i \langle n_i = 1 | A^+ | \rangle \Gamma_i^+$  in the interaction  $H'_{t,b}$  (7), and, subsequently, the operator  $A$  by  $\sum_i \langle n_i = 1 | A^+ | \rangle \Gamma_i$ . Here  $\langle n_i = 1 | A^+ | \rangle$  represents the matrix element of the operator  $A^+$  between the closed shell state and the state corresponding to the  $i$ th root in the TDA

$$H_{p,v} = - \sum_i A_i (A^+ \Gamma_i + A \Gamma_i^+) \quad (11)$$

Thus, the coupling strengths  $A_i$  are given by the product of the interaction strength  $V$  times the TDA matrix element of the operator  $A^+$ , i.e.

$$A_i = \begin{cases} V(2\Omega)^{\frac{1}{2}} & (i = 1) \\ 0 & (i = 2, 3, \dots, 2\Omega). \end{cases} \quad (12)$$

This value of  $A_i$  is also given<sup>1)</sup> by the matrix element of  $H'_{t,b}$  between any particle-hole state  $c_{m1}^+ c_{m\bar{1}} | \rangle$  and the TDA state  $| n_i = 1 \rangle$ ,

$$\langle n_i = 1 | H'_{t,b} c_{m1}^+ c_{m\bar{1}} | \rangle = -V \langle n_i = 1 | A^+ | \rangle = -A_i \quad (12')$$

In the present case, the non-adiabatic phonons are uncoupled from states involving, for instance, only the adiabatic phonons [since the corresponding coupling strengths (12) vanish and the energy denominators do not]. In the following, we only treat states involving the adiabatic phonon, and thus we drop the corresponding sub-index  $i = 1$ .

The sum of terms (7), (9) and (11) constitutes the field Hamiltonian  $H_f$ . One must supplement this Hamiltonian with the diagrammatic restrictions listed in sect. 1. In order to perform the diagrammatic perturbation treatment, we divide  $H_f$  in two parts

$$H_f = H_0 + H', \quad H_0 = H'_{s,p} + H_b, \quad H' = H'_{t,b} + H_{p,v}. \quad (13)$$

The zero-order term  $H_0$  includes both systems of independent particles and phonons. These two degrees of freedom are assumed to be independent, i.e.,

$$[\Gamma_i^+, c_{m1}] = [\Gamma_i, c_{m\bar{1}}^+] = 0. \quad (14)$$

The vertices corresponding to  $H'$  are given in figs. 1b and c. The overcompleteness implicit in the product basis is corrected through the perturbative treatment of  $H'$ .

If a given diagram is included, any other diagram<sup>1)</sup>, which is obtained from this one through a time permutation of the vertices, must also be taken into account,

<sup>†</sup> Throughout this paper, the TDA is identical to the RPA, because the Hamiltonians that we use [such as (1)] conserve the number of particle-hole pairs. Consequently, they do not induce ground-state correlations.

unless the new diagram is equivalent to the initial one and/or it contains vertices different from those of figs. 1b and c.

Each diagram is computed according to the usual rules <sup>6)</sup>. The numerator is given by the product of the vertices appearing in the diagram. The sign of the vertices is discussed in fig. 1. There is an additional minus sign for each crossing of fermion lines. The requirement of antisymmetry between fermion lines and of symmetry between phonon lines is neglected for intermediate states. However, there is a factor  $(n!)^{-\frac{1}{2}}$  for each subset of  $n$  identical phonons in initial and final states. Unlinked diagrams are neglected.

The denominators of each diagram are calculated according to appendix B, where the Brillouin-Wigner <sup>7)</sup> and Bloch-Horowitz <sup>8)</sup> perturbation treatments are briefly reviewed.

If  $\varepsilon$  and  $V\Omega$  are of the same order of magnitude [which we take to be  $O(1)$ ], a given diagram contributes to a definite order in powers of  $\Omega^{-1}$ . Each vertex 1b yields a factor  $V = O(\Omega^{-1})$ ; each vertex 1c, a factor  $A = O(\Omega^{-\frac{1}{2}})$ ; and each independent summation over single-particle states, a factor  $2\Omega$ .

The fermion particle-hole transition operator  $Q$  must be replaced, within the field treatment, by an operator including both particle and boson terms (fig 3). For instance, the operator

$$Q = \sum_m q_m c_{m1}^+ c_{m\bar{1}}, \quad (15)$$

is written as

$$Q_f = \sum_i \langle n_i = 1 | Q | \rangle \Gamma_i^+ + Q. \quad (16)$$

One must extend the definition of a bubble, which now includes processes in which the particle-hole pair is created by the operator  $Q$  and destroyed by  $H'$  (13), or conversely (fig. 2d).

Other operators, such as the one-particle creation or annihilation operators, which are used in one-particle transfer processes, do not have a boson image.

## 2.1 THE ONE-PHONON STATE

Any graph generated by the vertices of figs. 1b and c, and contributing to the energy of the one-phonon state, involves at least a bubble. Therefore, there are no corrections to the TDA energies (10), which coincide with the energies of the fermion treatment (see appendix A).

Similarly, there is a single diagram corresponding to the transition matrix element of the operator  $A^+$  between the closed shell and the one-phonon state. Fig. 3b represents this diagram. The TDA value of the vertex 3b and, thus, the value of the transition matrix element is

$$\langle n = 1 | A^+ | \rangle = (2\Omega)^{\frac{1}{2}} \quad (17)$$

The one-body transfer operators have no corresponding phonon vertices. Fig 4

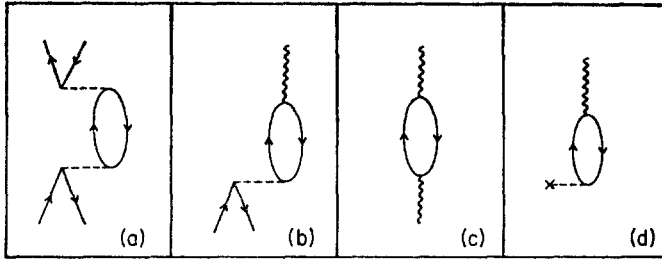


Fig. 2 Any diagram containing these processes (bubbles) as part of it, must be disregarded. The cross at the beginning of fig. (d) denotes a particle-hole operator.

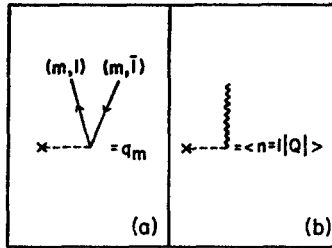


Fig. 3. The two vertices corresponding to a particle-hole operator [eq. (16)]

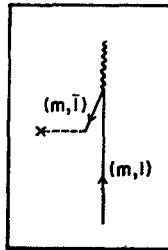


Fig. 4 The diagram representing the population of a phonon state by a single-capture reaction from the system with one more particle than the closed shell.

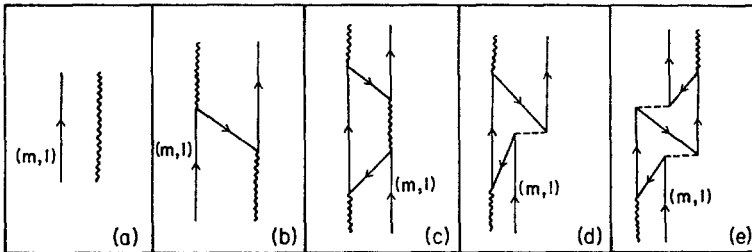


Fig. 5. Energy diagrams for the particle-phonon state.

represents the only possible diagram<sup>†</sup> representing the capture of a particle from below the Fermi sea, and which is in the same  $m$ -state as the initial particle state  $|m, 1\rangle = c_{m1}^+|\rangle$ . Its contribution is

$$\langle n = 1 | c_{m1}^- | m, 1 \rangle = \frac{A}{\omega - \varepsilon} = -(2\Omega)^{-\frac{1}{2}}. \quad (18)$$

The absolute value of this quantity is also the TDA amplitude of a particle-hole state in the normal mode. Thus, the results of the present subsection do not go beyond (but do not contradict) the values given by the TDA, which already agree with those of appendix A.

## 2.2 THE PARTICLE-PHONON STATES

More interesting from the point of view of the field theory are the particle-phonon states

In the odd nucleus, the diagram 5b contributes to the energy of the state  $|n = 1; m, 1\rangle$  with the value

$$- \frac{A^2}{\omega - \varepsilon} = V, \quad (19)$$

which is of the order  $\Omega^{-1}$  with respect to the zero-order excitation energy  $\omega$ . The sum  $\omega + V$  (graphs 5a and 5b) is also the exact excitation energy (appendix A). However, unlike the one-phonon case, many other diagrams are possible. For instance, graphs 5c and 5d represent the contributions of order  $\Omega^{-2}$ . In order to calculate all the higher-order perturbation terms, it is convenient to use the Brillouin-Wigner perturbation expansion<sup>††</sup> (appendix B). We thus take into account only diagrams in which the initial state is not an intermediate state, and we obtain eq. (B.12) for the energy. Graphs 5a, 5b, 5d, 5e, . . . yield a series expansion for  $W(E)$ , namely

$$\begin{aligned} W(E) &= \omega - \frac{A^2}{E - \varepsilon} - \frac{A^2 V}{(E - \varepsilon)^2} - \frac{A^2 V^2}{(E - \varepsilon)^3} + \dots \\ &= \omega - \frac{A^2}{E - \varepsilon - V} = \omega - \frac{2\Omega V^2}{E - \varepsilon - V}. \end{aligned} \quad (20)$$

Eqs. (B.12) and (20) yield the two energies<sup>†††</sup>

$$E = \begin{cases} \omega + V \\ \varepsilon. \end{cases} \quad (21)$$

Both roots coincide with the exact energies. The second one  $\varepsilon$  is also the unperturbed excitation energy of the intermediate two-particle, one-hole state.

<sup>†</sup> The diagram of fig. 4 also represents the population of a non-adiabatic phonon. In this case, both the numerator and denominator vanish [eqs. (10) and (12)]. The value of the diagram has to be obtained as a limiting process, such as the one outlined at the end of sect. 3 for  $\varepsilon_a \rightarrow \varepsilon_b$ .

<sup>††</sup> The Rayleigh-Schrodinger perturbation expansion has been used in (18) and (19).

<sup>†††</sup> The zero point for the energies in the odd system is placed at  $\frac{1}{2}\varepsilon$ .

In a Feynman diagrammatic expansion of the residual nuclear interaction, intermediate states may violate the Pauli principle. However, for a given diagram in which two fermion lines are simultaneously in the same single-particle state, there is another diagram in which the corresponding end points are interchanged. This second diagram cancels the first one. For instance, in fig. 6a, one of the possible intermediate states has  $(m', 1) = (m_0, 1)$ . This intermediate state cannot exist in the presence of an odd-nucleon in the single-particle state  $(m_0, 1)$ . There is another diagram (fig. 6b) in which the two particle lines of graph 6a are exchanged. The crossing of these two lines introduces a minus sign which cancels the component  $m' = m_0$  of the diagram 6a, which violates the Pauli principle.

The diagrams 6a and 6b are members of two subsets of graphs which are replaced by diagrams 6c and 6d, respectively, within the field treatment of the residual interaction. The effect of the process <sup>5)</sup> represented in fig. 6d amounts to subtracting a component which should not be present in the initial state (fig. 6c).

Ordinary perturbation theory would predict a negative second-order contribution  $\Delta E$  to the energy of the particle-phonon state, since the difference  $\omega - \varepsilon$  between the unperturbed energies of the initial and intermediate states is negative, and 6d is the only second-order contribution:

$$\begin{aligned} \Delta E = & \langle n = 1; m_0, 1 | H_{p,v} | m_1, 1; m_1, \bar{1}; m_0, 1 \rangle \\ & \times \langle m_1, 1; m_1, \bar{1}, m_0, 1 | H_{p,v} | n = 1; m_0, 1 \rangle / (\omega - \varepsilon). \end{aligned} \quad (22)$$

However, diagram 6d (= 5b) gives the positive value (19). The second-order contributions which would make  $\Delta E < 0$  involve bubbles (fig. 6e). Since these contributions are to be neglected, the correction becomes positive. This does not contradict eq. (22) since the numerator on the right-hand side is positive only for an Hermitian Hamiltonian. Or, although the Hamiltonian (13) appears to be so, it becomes highly non-Hermitian when complemented with the rule concerning the bubbles.

The square of the amplitude of the initial unperturbed state in the perturbed wave function is given by eq. (B 13), which in the present case yields

$$|\xi(n = 1, m, 1)|^2 = (1 - \partial W / \partial E)^{-1} = \left\{ 1 - \frac{2\Omega V^2}{4\Omega^2 V^2} \right\}^{-1} = \frac{2\Omega}{2\Omega - 1} > 1 \quad (23)$$

The fact that the modulus of an amplitude is larger than one is a further consequence of the non-Hermitian character of the Hamiltonian. The result (23) is consistent with the previous interpretation of diagram 6d since the corresponding contribution eliminates from the phonon a component which should not be there (because of the Pauli principle), the amplitudes of the remaining components should increase. According to eq. (23) they increase from  $(2\Omega)^{-\frac{1}{2}}$  (unperturbed phonon) to  $(2\Omega - 1)^{-\frac{1}{2}}$ . One can immediately see that this last one is the correct amplitude of a single particle-hole state if the blocking due to the presence of the odd particle is taken into account in the normal mode.

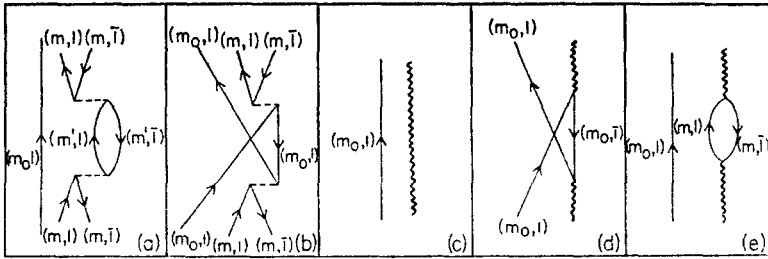


Fig 6 Graphs (a) and (b) represent sections of two Feynman diagrams which, together, take into account the antisymmetry of the intermediate state. Successive interactions between the particle  $(m, l)$  and  $(m, \bar{l})$  lines give rise to two series of graphs which are obtained in field theory by substituting (c) and (d) in the corresponding sections. Diagram (e) is forbidden.

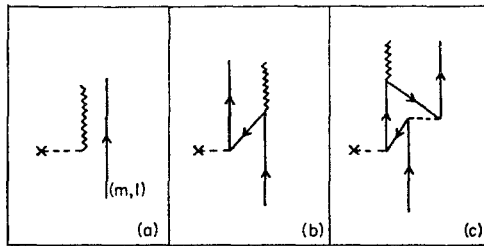


Fig 7 Diagrams corresponding to the inelastic excitation process in the odd system

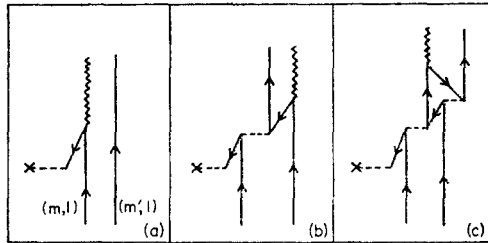


Fig 8 Diagrams corresponding to the one-body capture process populating the particle-phonon state.

We proceed to calculate the transition matrix element of the operator  $A_F^+$  [eqs (15)-(17)] from the single-particle state  $(m, l)$  to the particle-phonon state  $|n = 1, m, l\rangle$ . We apply eq (B.8). Since there is no correction to the energy of the single-particle state, the derivative  $\partial W/\partial E$  vanishes for the initial state and the corresponding amplitude equals one. The amplitude of the final state is the square root of (23).

The effective matrix element in (B.9) represents the contribution of all diagrams such that the initial and final states do not appear as intermediate states. The dia-

grams of fig 7 yield the series expansion

$$\begin{aligned} \langle n = 1; m, 1 | A_f^+ | m, 1 \rangle_e &= \langle n = 1 | A^+ | \rangle + \frac{A}{E - \varepsilon} + \frac{AV}{(E - \varepsilon)^2} + \dots \quad (24) \\ &= (2\Omega)^{\frac{1}{2}} + A/(E - \varepsilon - V) \\ &= (2\Omega)^{\frac{1}{2}}(1 - 1/2\Omega) \end{aligned}$$

The total matrix element is given by the product of the amplitude  $\xi(n = 1; m, 1)$  times the right-hand side of eq. (24), namely

$$\langle n = 1, m, 1 | A_f^+ | m, 1 \rangle = (2\Omega - 1)^{\frac{1}{2}}, \quad (25)$$

which again expresses the fact that one of the  $2\Omega$  particle-hole components is blocked in the odd system.

The result (18) concerning the capture of a nucleon in the state  $(m, \bar{1})$ , in a process populating the one-phonon state, is also modified by the presence of the odd particle

The initial state  $|m, 1, m', 1\rangle = c_{m1}^+ c_{m'1}^+ | \rangle$  represents now a nucleus with two particles above the closed shell, which also has an amplitude equal to one. The final state is the particle-phonon state  $|n = 1, m', 1\rangle$ . The diagrams of fig. 8 contribute to the effective matrix element in (B.9)

$$\begin{aligned} \langle n = 1; m', 1 | c_{m\bar{1}} | m, 1; m', 1 \rangle_e &= \frac{A}{E - \varepsilon} + \frac{AV}{(E - \varepsilon)^2} + \frac{AV^2}{(E - \varepsilon)^3} + \dots \\ &= \frac{A}{E - \varepsilon - V} = -(2\Omega)^{-\frac{1}{2}} \quad (26) \end{aligned}$$

The total transition matrix element is

$$\langle n = 1; m', 1 | c_{m\bar{1}} | m, 1; m', 1 \rangle = - \left( \frac{2\Omega}{2\Omega - 1} \right)^{\frac{1}{2}} \left( \frac{1}{2\Omega} \right)^{\frac{1}{2}} = -(2\Omega - 1)^{-\frac{1}{2}}, \quad (27)$$

which corresponds to the amplitude of the particle-hole pair  $(m\bar{1}, m\bar{1})$  in the one-phonon state if the single-particle state  $(m', 1)$  is blocked.

### 2.3. THE TWO-PHONON STATE

The number of graphs to be included in the present case increases considerably. However, the calculation may still be carried out to all orders of perturbation theory. With this aim, we classify the diagrams according to the number of independent summations over single-particle states. In the first group, we consider graphs containing a single summation. Their contribution can be cast into a series expansion, which may be inferred from diagrams 9a-9f. Their values are, respectively,

$$\begin{aligned} & - \frac{8A^4\Omega}{(E - \varepsilon - \omega)^2(E - 2\varepsilon)} + \frac{16A^6\Omega}{(E - \varepsilon - \omega)^3(E - 2\varepsilon)^2} - \frac{16A^4V}{(E - \varepsilon - \omega)^2(E - 2\varepsilon)^2} \\ & - \frac{32A^8\Omega}{(E - \varepsilon - \omega)^4(E - 2\varepsilon)^3} + \frac{64A^6V\Omega}{(E - \varepsilon - \omega)^3(E - 2\varepsilon)^3} - \frac{32A^4V^2\Omega}{(E - \varepsilon - \omega)^2(E - 2\varepsilon)^3} \\ & = B(1 - b + b^2) \rightarrow B/(1 + b), \quad (28) \end{aligned}$$

where

$$B = -\frac{8\Lambda^4\Omega}{(E-\varepsilon-\omega)^2(E-2\varepsilon)} = -\frac{32V^4\Omega^3}{(E-2\varepsilon+2V\Omega)^2(E-2\varepsilon)},$$

$$b = \frac{2\Lambda^2}{(E-\varepsilon-\omega)(E-2\varepsilon)} - \frac{2V}{E-2\varepsilon} = -\frac{2V}{E-2\varepsilon+2V\Omega}. \quad (29)$$

The contribution of diagrams containing a double summation over intermediate single-particle states may be cast into another series which is related to (28). The

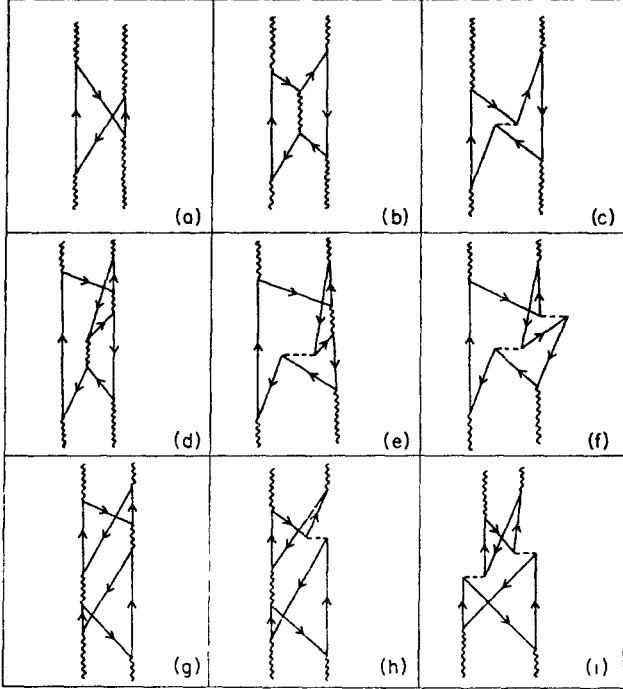


Fig 9 Graphs describing the interaction between two phonons

first order term of this series is obtained from diagrams 9g-9i, which yield, respectively,

$$\frac{32\Lambda^8\Omega^2}{(E-\varepsilon-\omega)^4(E-2\varepsilon)^3} - \frac{32\Lambda^6V\Omega^2(3E-4\varepsilon-2\omega)}{(E-\varepsilon-\omega)^4(E-2\varepsilon)^3} + \frac{32\Lambda^4V^2\Omega^2}{(E-\varepsilon-\omega)^2(E-2\varepsilon)^3} = -Bc, \quad (30)$$

where

$$c = \frac{4V^2\Omega(E-2\varepsilon-2V\Omega)}{(E-2\varepsilon+2V\Omega)^2(E-2\varepsilon)}.$$

We may check that subsequent terms of this second series yield

$$-Bc(1-2b+3b^2 \dots) = -Bc/(1+b)^2. \quad (32)$$

The terms containing three independent summations give the contributions

$$Bc^2(1-3b+6b^2+\dots) = Bc^2/(1+b)^3. \quad (33)$$

The contributions (28), (32) and (33), corresponding to the first three groups of diagrams, are the first three terms of a series which also may be added up. Thus, one obtains for  $W(E)$

$$\begin{aligned} W(E) &= 2\omega + \frac{B}{1+b} - \frac{Bc}{(1+b)^2} + \frac{Bc^2}{(1+b)^3} + \dots = 2\omega + \frac{B}{1+b+c} \\ &= 2\varepsilon - 4V\Omega - 32\Omega^3 V^4 / [(E-2\varepsilon)^3 + 2V(E-2\varepsilon)^2(2\Omega-1) \\ &\quad + 4V^2\Omega^2(E-2\varepsilon) - 8V^3\Omega^2] \end{aligned} \quad (34)$$

Eq (B 12) has now four roots, namely

$$E = \begin{cases} 2\omega + 2V \\ \omega + \varepsilon & \text{(double root)} \\ 2\varepsilon \end{cases} \quad (35)$$

The first eigenvalue lies in the neighbourhood of the energy of the two-phonon state. It is equal to the exact solution (appendix A) and it exhibits an anharmonicity in the collective vibrational motion that can be expressed as a phonon-phonon interaction energy of magnitude  $2V$ . The other two solutions are the unperturbed energies of the intermediate states appearing in fig 9†. They correspond to spurious states

### 3. Two single-particle degenerate excitations and the particle-hole monopole interaction

The general validity of the results obtained in the previous section may be restricted because of the special degeneracies there involved. In order to ascertain which results remain valid when these degeneracies are (at least partially) lifted, we assume that the states belonging to one of the two single-particle energies split in two subsets with equal degeneracy  $\Omega$ . Thus, there appear two single-particle excitation energies,  $\varepsilon_a^{(0)}$  and  $\varepsilon_b^{(0)}$ . We denote by  $a_{m\sigma}^+$  and  $b_{m\sigma}^+$  the corresponding fermion creation operators. The single-particle quantum numbers  $m$  and  $\sigma$  have the same meaning as before. The particles are coupled by the same interaction (3). Similarly to (4), we define

$$\begin{aligned} A_a^+ &= \sum_m a_{m1}^+ a_{m\bar{1}}, & A_b^+ &= \sum_m b_{m1}^+ b_{m\bar{1}}, \\ N_{a1} &= \sum_m a_{m1}^+ a_{m1}, & N_{b1} &= \sum_m b_{m1}^+ b_{m1}, \\ N_{a\bar{1}} &= \sum_m a_{m\bar{1}}^+ a_{m\bar{1}}, & N_{b\bar{1}} &= \sum_m b_{m\bar{1}}^+ b_{m\bar{1}}, \end{aligned} \quad (36)$$

† The fact that the root  $2\varepsilon$  also corresponds to an exact excitation energy is only due to the special degeneracies of the model. The same is true for the energy  $\varepsilon$  in eq. (21)

where the summation  $\sum$  extends over  $\Omega$ -states.

Using the same arguments as in sect. 2, we obtain the field Hamiltonian. The fermion components are

$$H'_{s\ p} = \frac{1}{2}\varepsilon_a(N_{a1} + N_{a\bar{1}}) + \frac{1}{2}\varepsilon_b(N_{b1} + N_{b\bar{1}}), \quad H'_{t\ b} = -VA^+A, \quad (37)$$

where the Hartree-Fock particle-hole excitation energies are

$$\varepsilon_a = \varepsilon_a^{(0)} + V, \quad \varepsilon_b = \varepsilon_b^{(0)} + V, \quad (38)$$

and  $A^+$  is the sum  $A_a^+ + A_b^+$  given in eq. (4)

The TDA dispersion relation is

$$\frac{1}{V} = \frac{\Omega}{\varepsilon_a - \omega} + \frac{\Omega}{\varepsilon_b - \omega}, \quad (39)$$

which has two roots  $\omega_\alpha$  and  $\omega_\beta$ :

$$\omega_{\alpha/\beta} = \frac{1}{2}(\varepsilon_a + \varepsilon_b - 2V\Omega) \mp \frac{1}{2}[(\varepsilon_b - \varepsilon_a)^2 + 4V^2\Omega^2]^{\frac{1}{2}}. \quad (40)$$

The two creation operators for the TDA phonons are defined as

$$\alpha^+ = \lambda A_a^+ + \mu A_b^+, \quad \beta^+ = -\mu A_a^+ + \lambda A_b^+, \quad (41)$$

where

$$\lambda = \frac{A_\alpha}{\varepsilon_a - \omega_\alpha} = \frac{A_\beta}{\varepsilon_b - \omega_\beta}, \quad \mu = \frac{A_\alpha}{\varepsilon_b - \omega_\alpha} = -\frac{A_\beta}{\varepsilon_a - \omega_\beta}. \quad (42)$$

The normalization coefficients  $A_\alpha$  and  $A_\beta$  are given by

$$A_{\alpha/\beta} = \left\{ \frac{\Omega}{(\varepsilon_a - \omega_{\alpha/\beta})^2} + \frac{\Omega}{(\varepsilon_b - \omega_{\alpha/\beta})^2} \right\}^{-\frac{1}{2}} = V \left\{ \Omega \pm \frac{2V\Omega^2}{[(\varepsilon_a - \varepsilon_b)^2 + 4V^2\Omega^2]^{\frac{1}{2}}} \right\}^{\frac{1}{2}}. \quad (43)$$

The independent boson Hamiltonian and the particle-vibration interaction are, respectively,

$$\begin{aligned} H_b &= \omega_\alpha \alpha^+ \alpha + \omega_\beta \beta^+ \beta, \\ H_p &= -V \langle n_\alpha = 1 | A^+ | \rangle [\alpha^+ A + A^+ \alpha] - V \langle n_\beta = 1 | A^+ | \rangle [\beta^+ A + A^+ \beta] \\ &= -A_\alpha (\alpha^+ A + A^+ \alpha) - A_\beta (\beta^+ A + A^+ \beta) \end{aligned} \quad (44)$$

The field operator for the inelastic scattering is written

$$A_i^+ = A^+ + \langle n_\alpha = 1 | A^+ | \rangle \alpha^+ + \langle n_\beta = 1 | A^+ | \rangle \beta^+ = A^+ + \frac{A_\alpha}{V} \alpha^+ + \frac{A_\beta}{V} \beta^+. \quad (45)$$

We concentrate the discussion of this section on the odd system. We choose as basic subset of states the pair

$$|\alpha\rangle = \alpha^+ a_{m1}^+ | \rangle, \quad |\beta\rangle = \beta^+ a_{m1}^+ | \rangle, \quad (46)$$

with unperturbed excitation energies  $\omega_x$  and  $\omega_\beta$ , respectively<sup>†</sup> We use the Bloch-Horowitz formalism in order to simultaneously perturb these two states (appendix B) We construct an effective Hamiltonian within the two-dimensional space (46) Each matrix element is obtained from a set of diagrams analogous to the ones in fig. 5, where the odd particle has been created by the operator  $a_{m_1}^\dagger$ , and the initial ( $\zeta$ ) and final ( $\rho$ ) phonons may be either  $\alpha$  or  $\beta$ . One thus obtains

$$\begin{aligned} W_{\rho\zeta}(E) &= \omega_\rho \delta_{\rho\zeta} - \frac{A_\rho A_\zeta}{E - \varepsilon_a} - \frac{A_\rho A_\zeta V}{(E - \varepsilon_a)^2} - \frac{A_\rho A_\zeta V^2}{(E - \varepsilon_a)^3} + \dots \\ &= \omega_\rho \delta_{\rho\zeta} - A_\rho A_\zeta / (E - \varepsilon_a - V). \end{aligned} \quad (47)$$

The determinantal eq (B.5) can now be written as

$$\begin{vmatrix} W_{\alpha\alpha}(E) - E & W_{\alpha\beta}(E) \\ W_{\beta\alpha}(E) & W_{\beta\beta}(E) - E \end{vmatrix} \\ = [E^2 - E(\varepsilon_a + \varepsilon_b - 2V\Omega + V) + \varepsilon_a \varepsilon_b - (\varepsilon_a + \varepsilon_b)V\Omega + V\varepsilon_b](E - \varepsilon_a)/(E - \varepsilon_a - V) = 0. \quad (48)$$

The root  $E = \varepsilon_a$  corresponds to the unperturbed energy of the intermediate state [as in (21) and (35)]. The first factor on the right-hand side of (48) has the same roots as the secular equation in the fermion treatment (appendix A), namely

$$E_\pm = \frac{1}{2}(\varepsilon_a + \varepsilon_b - 2V\Omega + V) \pm \frac{1}{2}[(\varepsilon_a - \varepsilon_b + V)^2 + 4V^2\Omega(\Omega - 1)]^{\frac{1}{2}}. \quad (49)$$

The Schrodinger equation in the basic space and the normalization condition determine the amplitudes

$$\xi_\beta / \xi_\alpha = (E - W_{\alpha\alpha}) / W_{\alpha\beta}; \quad 1 = \left[ \xi_\alpha^2 \left( 1 - \frac{\partial W_{\alpha\alpha}}{\partial E} \right) + \xi_\beta^2 \left( 1 - \frac{\partial W_{\beta\beta}}{\partial E} \right) - 2\xi_\alpha \xi_\beta \frac{\partial W_{\alpha\beta}}{\partial E} \right], \quad (50)$$

and yield

$$\begin{aligned} \xi_\alpha &= [(E - W_{\beta\beta})(E - \varepsilon_a - V)(E - \varepsilon_a + V\Omega - V) / (E - \varepsilon_a)]^{\frac{1}{2}} \eta, \\ \xi_\beta &= -[(E - W_{\alpha\alpha})(E - \varepsilon_a - V)(E - \varepsilon_a + V\Omega - V) / (E - \varepsilon_a)]^{\frac{1}{2}} \eta, \end{aligned} \quad (51)$$

where

$$\eta = [(E - \varepsilon_a + V\Omega - V)^2 + V^2\Omega(\Omega - 1)]^{-\frac{1}{2}} \quad (52)$$

The phases correspond to the  $\pm$  sign in eq. (49). The ratio of the two amplitudes (51) is negative for both roots. This is due to the fact that the two eigenstates are orthonormal within the complete set of states, but not within the basic subset.

The matrix element corresponding to the inelastic process is calculated using ex-

<sup>†</sup> The zero point of the energy has been placed at  $\frac{1}{2}\varepsilon_a$ . It is also assumed that  $\varepsilon_b > \varepsilon_a$

pression (45) for the operator  $A_f^+$ :

$$\begin{aligned}\langle \psi | A_f^+ | m_a, 1 \rangle &= \xi_\alpha \langle \alpha | A_f^+ | m_a, 1 \rangle_e + \xi_\beta \langle \beta | A_f^+ | m_a, 1 \rangle_e \\ &= (\xi_\alpha A_\alpha + \xi_\beta A_\beta)(E - \varepsilon_a) / V(E - \varepsilon_a - V) \\ &= -\eta(E - \varepsilon_a) \Omega^{\frac{1}{2}},\end{aligned}\quad (53)$$

where the effective matrix elements are obtained from a graphical expansion similar to the one represented in fig. 7.

Similar expressions hold for the one-body capture process. Using fig. 8, we obtain

$$\begin{aligned}\langle \psi | b_{m\bar{1}} | m'_b, 1, m_a, 1 \rangle &= \xi_\alpha \langle \alpha | b_{m\bar{1}} | m'_b, 1; m_a, 1 \rangle_e + \xi_\beta \langle \beta | b_{m\bar{1}} | m'_b, 1; m_a, 1 \rangle_e \\ &= (\xi_\alpha A_\alpha + \xi_\beta A_\beta)(E - \varepsilon_a) / (E - \varepsilon_b)(E - \varepsilon_a - V) \\ &= -[(E - \varepsilon_a) / (E - \varepsilon_b)] V \eta \Omega^{\frac{1}{2}}.\end{aligned}\quad (54)$$

Expressions (49), (53) and (54) yield (21), (25) and (27), respectively, in the limit  $\varepsilon_a \rightarrow \varepsilon_b$ . Moreover, they agree with the corresponding values which are derived in appendix A through the fermion calculation. Thus, we conclude that the exact agreement of sect 2 between the results corresponding to the fermion and field treatments of the residual interactions, extends also to the case in which the degeneracy of sect 2 is (at least partially) lifted.

Since for  $\varepsilon_a \rightarrow \varepsilon_b$  the higher energy phonon (40) becomes non-adiabatic, we may also obtain from the present section the properties of the non-adiabatic phonons which are present in the case of degenerate particle-hole excitations

Another interesting limit occurs for  $\Omega = 1$ . In this case, the degeneracies of the particle-hole spectrum are completely removed, and there is a single two-particle, one-hole state such that the odd-particle is in the  $a_1^+ | \rangle$  state<sup>†</sup>.

The state is represented by  $b_1^+ b_{\bar{1}} a_1^+ | \rangle$  in the fermion treatment. However, there are two roots (49) present in the field treatment, namely

$$E = \begin{cases} \varepsilon_b - V \\ \varepsilon_a. \end{cases}\quad (49')$$

The upper root corresponds to the real energy, while the lower one yields the unperturbed energy  $\varepsilon_a$ . According to eqs. (53) and (54), the matrix elements of the operators  $A_f^+$  and  $b_{\bar{1}}$ , corresponding to the population of the upper root, have the (correct) value of one. In order to calculate the transition to the spurious root we note that the three quantities  $E - \varepsilon_a$ ,  $E - \varepsilon_a + V\Omega - V$  and  $V^2\Omega(\Omega - 1)$  are in this case of order  $\delta \equiv \Omega - 1$ . Thus, the matrix elements (52) and (54) are of  $O(\delta^{\frac{1}{2}})$  for the lowest root and, therefore, vanish for  $\delta \rightarrow 0$ . In spite of the fact that there is a spurious state in our basis (46), this state cannot be populated, and the results of the fermion calculation are always reproduced.

<sup>†</sup> We drop here the subindex  $m$ , which is redundant for  $\Omega = 1$ .

#### 4. Particle-hole degenerate excitations and the simultaneous treatment of a particle-hole and a pairing monopole residual interaction

The fermion Hamiltonian is now written<sup>†</sup>

$$H = H_{s,p} + H_{t,b}, \quad H_{s,p} = \frac{1}{2}\varepsilon_0(N_1 + N_{\bar{1}}),$$

$$H_{t,b} = -\frac{1}{2}V(A_0^+ A_0 + A_0 A_0^+) - \frac{1}{2}G(A_1^+ A_1 + A_1 A_1^+ + A_{\bar{1}}^+ A_{\bar{1}} + A_{\bar{1}} A_{\bar{1}}^+), \quad (55)$$

where

$$A_0^+ = A^+ = \sum_m c_{m1}^+ c_{m\bar{1}}, \quad A_1^+ = \sum_{m>0} c_{m1}^+ c_{m\bar{1}}, \quad A_{\bar{1}}^+ = \sum_{m>0} c_{m\bar{1}}^+ c_{m1}. \quad (56)$$

The operators  $H_{s,p}$ ,  $N_\sigma$  and  $c_{m\sigma}^+$  have the same meaning as in sect 2, as well as the single-particle quantum numbers  $(m, \sigma)$ . The operator  $c_{m\sigma}^+$  is obtained from  $c_{m\sigma}^+$  through the time reversal transformation

The fermion treatment of the Hamiltonian (55) is carried out in appendix A. The procedure is simplified because the  $SP_4$  algebra may be applied<sup>9)</sup>

We proceed to calculate the field Hamiltonian. The single-particle basis satisfies the Hartree-Fock minimization condition as before.

In the present case, there is also a Hartree-Fock contribution to the single-particle excitation energy, namely

$$\varepsilon = \varepsilon_0 + V + G. \quad (57)$$

The single-particle term in the field treatment has the same expression as  $H'_{s,p}$  in eq. (7) with  $\varepsilon$  given by (57). The difference  $H_{s,p} - H'_{s,p}$  must be added to  $H_{t,b}$  in order to obtain the four-point interaction  $H'_{t,b}$  to be used in the field treatment. This amounts to subtracting from  $H_{t,b}$  the Hartree-Fock self-energy insertions. The vertices corresponding to  $H'_{t,b}$  are given in figs 10a and b.

$$H'_{s,p} = \frac{1}{2}\varepsilon(N_1 + N_{\bar{1}}), \quad H'_{t,b} = -VA_0^+ A_0 - G(A_1^+ A_1 + A_{\bar{1}}^+ A_{\bar{1}}). \quad (58)$$

Both particle-hole ( $\alpha = 0$ ) and pairing phonons ( $\alpha = 1, \bar{1}$ ) are present in the independent boson Hamiltonian<sup>††</sup>. As before, we treat only the adiabatic phonons

$$H_b = \sum_\alpha \omega_\alpha \Gamma_\alpha^+ \Gamma_\alpha, \quad \omega_0 = \varepsilon - 2V\Omega, \quad \omega_1 = \omega_{\bar{1}} = \varepsilon - G\Omega. \quad (59)$$

The particle-vibration interaction  $H_{p,v}$  is obtained replacing  $A_\alpha^+$  by  $\langle n_\alpha = 1 | A_\alpha^+ | \rangle \Gamma_\alpha^+$  in  $H'_{t,b}$ , where the matrix element of  $A_\alpha^+$  is obtained again in the TDA. We proceed similarly for the adjoint operator  $A_\alpha$ . Thus, we obtain

$$H_{p,v} = - \sum_\alpha A_\alpha (\Gamma_\alpha^+ A_\alpha + A_\alpha^+ \Gamma_\alpha), \quad (60)$$

where

$$A_\alpha = \begin{cases} V \langle n_0 = 1 | A_0^+ | \rangle = V(2\Omega)^{\frac{1}{2}}, & \alpha = 0 \\ G \langle n_1 = 1 | A_1^+ | \rangle = G(\Omega)^{\frac{1}{2}}, & \alpha = 1, \bar{1}. \end{cases} \quad (61)$$

The vertices corresponding to the interaction (60) are given in figs 10c and d.

<sup>†</sup> The interaction  $H_{t,b}$  has been symmetrized with respect to particles and holes as in (3)

<sup>††</sup> The label  $\alpha$  is the pair-transfer quantum number.

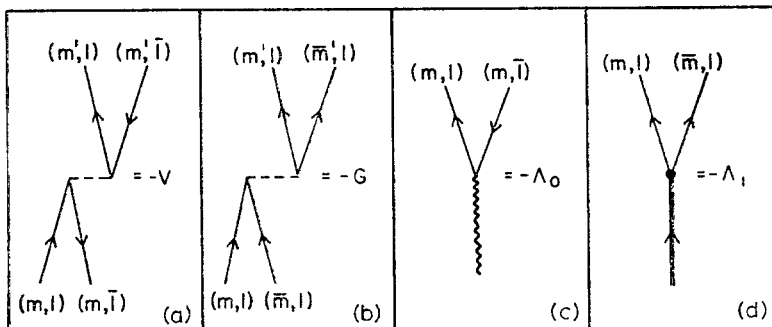


Fig 10 The vertices corresponding to the two-body interaction (58) and to the particle-vibration interaction (60). The sign of the vertices (a) and (c) is discussed in fig 1. The value of the vertices (b) and (d) is as indicated in the figure, when the particle line with  $m > 0$  appears to the left of the time-reversed state  $\bar{m} < 0$ , for all pairs arriving or leaving the vertex. A minus sign appears for each interchange of this order

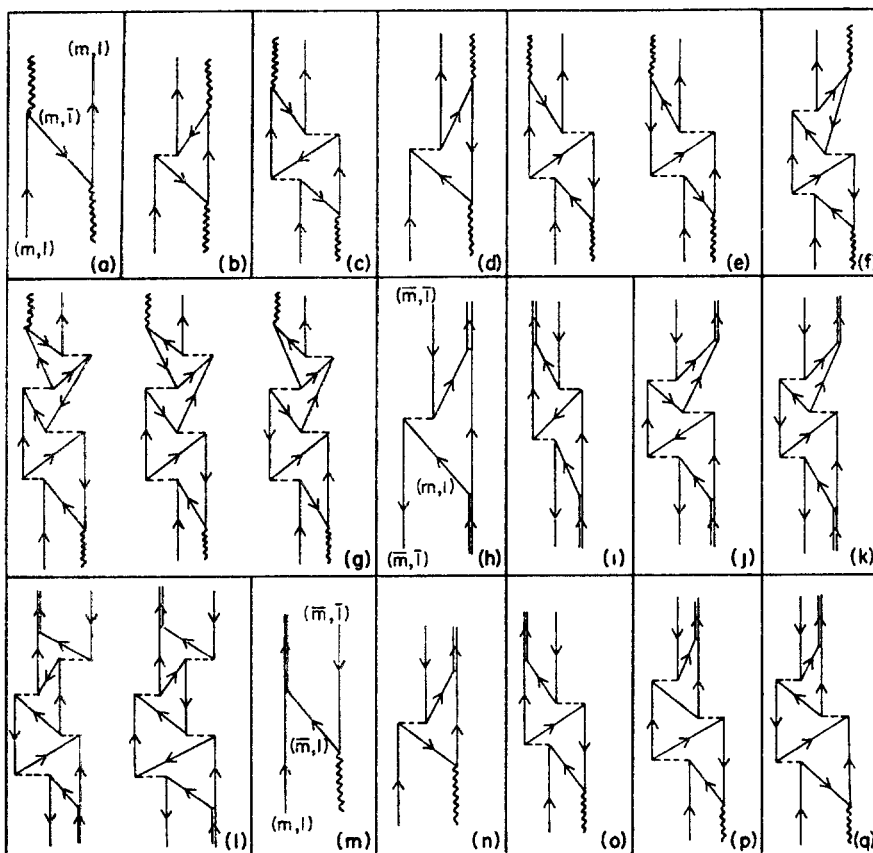


Fig 11 Graphs contributing to the matrix elements of the effective Hamiltonian to be used within the basic subset of states (62)

The field Hamiltonian  $H'_{s_p} + H'_{t_b} + H_b + H_{p_v}$ , always includes the diagrammatic restrictions listed in sect 1. In addition to the graphs containing the bubbles of fig. 2, diagrams in which a pair of particles (holes) in time reverse states are consecutively created and destroyed, must also be disregarded.

Our basic subset of particle-phonon states is given by the two states

$$|0\rangle = c_{m1}^+ \Gamma_0^+ | \rangle, \quad |1\rangle = c_{\bar{m}1} \Gamma_1^+ | \rangle, \quad (62)$$

with unperturbed energies  $\omega_0$  and  $\omega_1$ , respectively. The matrix elements of the effective Hamiltonian are obtained from fig. 11. Diagrams a-g belong to a double series which yields the diagonal term  $W_{00}(E)$ :

$$\begin{aligned} W_{00}(E) &= \omega_0 - \frac{A_0^2}{E-\varepsilon} - \frac{A_0^2 V}{(E-\varepsilon)^2} - \frac{A_0^2 V^2}{(E-\varepsilon)^3} \dots - \frac{A_0^2 G}{(E-\varepsilon)^2} - \frac{2A_0^2 G V}{(E-\varepsilon)^3} \dots \\ &\quad - \frac{A_0^2 G^2 V}{(E-\varepsilon)^4} - \frac{3A_0^2 G^2 V^2}{(E-\varepsilon)^5} - \dots \\ &= \omega_0 - \frac{A_0^2}{E-\varepsilon-V} - \frac{A_0^2 G}{(E-\varepsilon-V)^2} - \frac{A_0^2 G^2 V}{(E-\varepsilon)(E-\varepsilon-V)^3} \dots \\ &= \omega_0 - \frac{A_0^2}{E-\varepsilon-V} - \frac{A_0^2 G}{(E-\varepsilon-V)^2} \left/ \left( 1 - \frac{G V}{(E-\varepsilon)(E-\varepsilon-V)} \right) \right. \\ &= \varepsilon - 2V\Omega(E-\varepsilon)^2 / ((E-\varepsilon)(E-\varepsilon-V) - GV). \end{aligned} \quad (63)$$

The other diagonal matrix element,  $W_{11}(E)$ , is obtained from diagrams h-l

$$\begin{aligned} W_{11}(E) &= \omega_1 - \frac{A_1^2 V}{(E-\varepsilon)^2} - \frac{A_1^2 V^2}{(E-\varepsilon)^3} - \frac{A_1^2 V^3}{(E-\varepsilon)^4} \dots - \frac{A_1^2 V^2 G}{(E-\varepsilon)^4} - \frac{2A_1^2 V^3 G}{(E-\varepsilon)^5} - \dots \\ &= \omega_1 - \frac{A_1^2 V}{(E-\varepsilon)(E-\varepsilon-V)} - \frac{A_1^2 V^2 G}{(E-\varepsilon)^2(E-\varepsilon-V)^2} + \dots \\ &= \omega_1 - A_1^2 V / ((E-\varepsilon)(E-\varepsilon-V) - GV) \\ &= \varepsilon - G\Omega(E-\varepsilon)(E-\varepsilon-V) / ((E-\varepsilon)(E-\varepsilon-V) - GV). \end{aligned} \quad (64)$$

Finally, the non-diagonal matrix element  $W_{01}(E)$  is given by the series corresponding to diagrams m-q.

$$\begin{aligned} W_{01}(E) &= \frac{A_0 A_1}{E-\varepsilon} + \frac{A_0 A_1 V}{(E-\varepsilon)^2} + \dots + \frac{A_0 A_1 V G}{(E-\varepsilon)^3} + \frac{2A_0 A_1 V^2 G}{(E-\varepsilon)^4} + \dots \\ &= \frac{A_0 A_1}{E-\varepsilon-V} + \frac{A_0 A_1 V G}{(E-\varepsilon)(E-\varepsilon-V)^2} + \dots \\ &= A_0 A_1 (E-\varepsilon) / ((E-\varepsilon)(E-\varepsilon-V) - VG) \\ &= GV\Omega\sqrt{2}(E-\varepsilon) / ((E-\varepsilon)(E-\varepsilon-V) - VG) \end{aligned} \quad (65)$$

Equating to zero the determinant of the matrix  $W-E$ , we obtain the fourth-order equation

$$(E-\varepsilon)^2((E-\varepsilon)^2+(E-\varepsilon)(G\Omega+2V\Omega-V)-GV(1+\Omega-2\Omega^2))=0, \quad (66)$$

which has two roots corresponding to the unperturbed energy  $\varepsilon$  and two roots which coincide with those of the fermion treatment

$$E_{\pm} = \varepsilon - \frac{1}{2}(G\Omega + 2V\Omega - V) \pm \frac{1}{2}A, \quad (67)$$

where

$$A = ((G\Omega + 2V\Omega - V)^2 + 4VG(1 + \Omega - 2\Omega^2))^{\frac{1}{2}}. \quad (68)$$

The amplitudes of the basic states (62) are obtained using two equations similar to (50). They yield

$$\begin{aligned} \xi_0 &= \mp 2V/\{A^2 \pm A((2V-G)\Omega - V - 2V/\Omega)\}^{\frac{1}{2}}, \\ \xi_1 &= \frac{A \pm (G\Omega - 2V\Omega + V)}{\sqrt{2}\{A^2 \pm A((2V-G)\Omega - V - 2V/\Omega)\}^{\frac{1}{2}}} \end{aligned} \quad (69)$$

The diagrams contributing to the effective matrix element  $\langle 0|A_f^+|m, 1\rangle_e$  are essentially the same as those contributing to  $W_{00}(E)$ . For the fermion component  $A_0^+$  of the operator  $A_f^+$ , we must replace the initial phonon by the creation of a particle-hole pair (fig. 3a) in the diagrams 11a-g. Thus, the contribution of the fermion component  $A_0^+$  is  $(\omega_0 - W_{00})/A_0$ . The contribution of the phonon component of  $A_f^+$  is given by graph 7a,

$$\begin{aligned} \langle 0|A_f^+|m, 1\rangle_e &= \langle n_0 = 1|A_0^+| \rangle + (\omega_0 - W_{00})/A_0 = (2\Omega)^{\frac{1}{2}} + \frac{A_0}{E - \varepsilon - V} \\ &+ \frac{A_0 G(E - \varepsilon)}{(E - \varepsilon - V)((E - \varepsilon)(E - \varepsilon - V) - GV)} = \frac{(E - \varepsilon)^2 (2\Omega)^{\frac{1}{2}}}{(E - \varepsilon)(E - \varepsilon - V) - GV}. \end{aligned} \quad (70)$$

Similarly, the other effective matrix element is obtained through replacement of the initial phonon by the creation of the particle-hole pair in graphs 11m-q

$$\langle 1|A_f^+|m, 1\rangle_e = -W_{01}/A_0 = -G(\Omega)^{\frac{1}{2}}(E - \varepsilon)/((E - \varepsilon)(E - \varepsilon - V) - VG). \quad (71)$$

Therefore, the total matrix element for the inelastic scattering in the odd system is

$$\begin{aligned} \langle \psi|A_f^+|m, 1\rangle &= \xi_0 \langle 0|A_f^+|m, 1\rangle_e + \xi_1 \langle 1|A_f^+|m, 1\rangle_e \\ &= \mp \frac{1}{\sqrt{2}\Omega} \frac{G\Omega + 2V\Omega - V \mp A}{\{A^2 \pm A((2V-G)\Omega - V - 2V/\Omega)\}^{\frac{1}{2}}} \end{aligned} \quad (72)$$

which agrees with the result of the fermion calculation. If (72) is expanded in powers of  $\Omega^{-1}$ , one obtains the result of ref. <sup>2</sup>), where terms of order  $\Omega^{-1}$  and  $\Omega^{-2}$  were calculated using Rayleigh-Schrödinger perturbation expansion.

## 5. Conclusions

We have applied the nuclear field treatment <sup>1)</sup> to simple but physically meaningful models, which are exactly soluble both with the fermion and field treatments. The two methods give identical results, if they are carried out to all orders of perturbation theory.

The first model which is used consists of degenerate particle-hole excitations and a monopole particle-hole interaction (sect. 2). A detailed discussion of how the Pauli corrections are done within the field treatment is presented for the odd system. In addition to the energy, these Pauli corrections affect the transition rates and the norm of the states

The agreement of sect. 2 does not depend on the particular degeneracies of the model, as is shown in sect. 3, where this degeneracy is partially lifted

The basic set of states is also overcomplete, since we include both phonon and particle degrees of freedom. Even the proper states (to whom the initial and final states are restricted) are too many as compared with the states of the fermion treatment. An example of this situation is discussed in sect. 3, for the case of two non-degenerate particle-hole excitations. There are two proper particle-phonon states (with the particle in a fixed state  $\phi_a$ ), but there is a single two-particle one-hole antisymmetrized fermion state with the odd particle in the same state  $\phi_a$ . One of the states obtained in the field treatment has the correct properties, while all the matrix elements corresponding to transitions populating the second state vanish. In this way the field treatment also corrects for the overcompleteness of the basis. This overcompleteness is increased when pairing phonons are also included. However, the field treatment reproduces the exact results also in this case.

Discussions with A. Bohr, P. F. Bortignon, T. Marumori, B. R. Mottelson and R. P. J. Perazzo are gratefully acknowledged.

## Appendix A

### THE MATRIX ELEMENTS OF THE FERMION HAMILTONIAN

#### A.1 MONOPOLE PARTICLE-HOLE HAMILTONIAN

The operators  $A^+$ ,  $A$  and  $\frac{1}{2}(N_1 + N_{\bar{1}} - 2\Omega)$  obey the angular momentum commutation relations of  $I_+$ ,  $I_-$  and  $I_z$  (quasi-spin formalism), namely,

$$\begin{aligned} [A^+, A] &= N_1 + N_{\bar{1}} - 2\Omega, \\ [\frac{1}{2}(N_1 + N_{\bar{1}} - 2\Omega), A^+] &= A^+ \end{aligned} \quad (\text{A.1})$$

The smallest possible value of  $I_z$  is  $-\Omega$  which appears for  $N_{\bar{1}} = N_1 = 0$ . Therefore, the maximum value of  $I$  ( $I = \Omega$ ) occurs for closed-shell systems. For the neighbouring odd systems, the maximum value of  $I$  is  $\Omega - \frac{1}{2}$ . Unless specified we assume

one of these two representations, since the Hamiltonian is such that the states with largest  $I$  lie lowest in energy.

It is convenient to label the states of the even system by the number of particle-hole pairs,  $N = \frac{1}{2}(N_1 + N_{\bar{1}})$ . For the neighbouring odd nuclei we specify the quantum numbers  $(m, \sigma)$  of the odd particle or hole in addition to the number of particle-hole pairs  $N = \frac{1}{2}(N_1 + N_{\bar{1}})$ . In this representation, the operators  $N_1, N_{\bar{1}}$  are diagonal. The matrix elements of  $A^+$  are identical with those of  $I_+$ ,

$$\langle N' | A^+ | N \rangle = \delta_{N', N+1} [(2\Omega - N)(N + 1)]^{\frac{1}{2}},$$

$$\langle m', \sigma'; N' | A^+ | m, \sigma; N \rangle = \delta_{m, m'} \delta_{\sigma, \sigma'} \delta_{N', N+1} [(2\Omega - N - 1)(N + 1)]^{\frac{1}{2}}. \quad (\text{A } 2)$$

Using (A.2) we can evaluate the non-vanishing matrix elements of the monopole particle-hole interaction (7). For the degenerate particle-hole excitation model we have, in the closed-shell system

$$\langle N | H'_{i_b} | N \rangle = -VN(2\Omega - N + 1), \quad (\text{A.3})$$

which yields, in particular, the energies of the state with one and two phonons

$$\langle N = 1 | H | N = 1 \rangle = \omega = \varepsilon - 2V\Omega, \quad (\text{A } 4)$$

$$\langle N = 2 | H | N = 2 \rangle = 2\varepsilon - 4V\Omega + 2V = 2(\omega + V) \quad (\text{A } 5)$$

In the odd system,

$$\langle m, \sigma; N | H'_{i_b} | m, \sigma; N \rangle = -VN(2\Omega - N), \quad (\text{A.6})$$

and thus, the excitation energy of a particle-phonon state is

$$\langle m, \sigma; N = 1 | H | m, \sigma; N = 1 \rangle - \langle m, \sigma; N = 0 | H | m, \sigma; N = 0 \rangle = \varepsilon - 2V\Omega + V = \omega + V. \quad (\text{A } 7)$$

Using the representation  $I = \Omega - 1$  for the closed-shell system we obtain the unperturbed energy  $\varepsilon = \varepsilon_0 + V$  as the excitation energy of  $2\Omega - 1$  degenerate states. This same excitation energy is obtained in the odd system for  $I = \Omega - \frac{3}{2}$

The previous results can easily be extended to two degenerate particle-hole excitations. We now define the number of particle-hole pairs as  $N_c = \frac{1}{2}(N_{c1} + N_{c\bar{1}})$  with  $c = a$  or  $b$ , for the even system. For the odd system we assume that the particle is in a state  $a$ , so that  $N_a = \frac{1}{2}(N_{a1} + N_{a\bar{1}} - 1)$  and  $N_b = \frac{1}{2}(N_{b1} + N_{b\bar{1}})$ .

With a degeneracy  $\Omega$  for each excitation, the matrix elements of the operators  $A^+$  are

$$\langle N'_c | A_c^+ | N_c \rangle = \delta_{N'_c, N_c+1} [(\Omega - N_c)(N_c + 1)]^{\frac{1}{2}},$$

$$\langle m'_a, \sigma'; N'_a | A_a^+ | m_a, \sigma; N_a \rangle = \delta_{m_a, m'_a} \delta_{\sigma, \sigma'} \delta_{N'_a, N_a+1} [(\Omega - N_a - 1)(N_a + 1)]^{\frac{1}{2}},$$

$$\langle m'_a, \sigma'; N'_b | A_b^+ | m_a, \sigma; N_b \rangle = \delta_{m_a, m'_a} \delta_{\sigma, \sigma'} \delta_{N'_b, N_b+1} [(\Omega - N_b)(N_b + 1)]^{\frac{1}{2}}, \quad (\text{A.8})$$

and the matrix elements of the Hamiltonian (7) are

$$\begin{aligned}
 \langle N_a = 1, N_b = 0 | H'_{t_b} | N_a = 1, N_b = 0 \rangle &= \langle N_a = 0, N_b = 1 | H'_{t_b} | N_a = 0, N_b = 1 \rangle \\
 &= \langle N_a = 1, N_b = 0 | H'_{t_b} | N_a = 0, N_b = 1 \rangle = -V\Omega, \\
 \langle m_a, \sigma; N_a = 1, N_b = 0 | H'_{t_b} | m_a, \sigma; N_a = 1, N_b = 0 \rangle \\
 &= \langle m_a, \sigma; N_a = 1, N_b = 0 | H'_{t_b} | m_a, \sigma; N_a = 0, N_b = 1 \rangle = -V(\Omega-1), \\
 \langle m_a, \sigma; N_a = 0, N_b = 1 | H'_{t_b} | m_a, \sigma; N_a = 0, N_b = 1 \rangle &= -V\Omega.
 \end{aligned} \tag{A.9}$$

The secular equation yields the roots

$$\omega_{\alpha/\beta} = \frac{1}{2}(\varepsilon_a + \varepsilon_b) - V\Omega \pm \frac{1}{2}[(\varepsilon_a - \varepsilon_b)^2 + 4V^2\Omega^2]^{\frac{1}{2}}, \tag{A.10}$$

for the energy of the phonon and

$$E_{\pm} = \frac{1}{2}(\varepsilon_a + \varepsilon_b - 2V\Omega + V) \pm \frac{1}{2}[(\varepsilon_a - \varepsilon_b + V)^2 + 4V^2\Omega(\Omega-1)]^{\frac{1}{2}}. \tag{A.11}$$

for the energy of the odd system, while the amplitudes are given by

$$\begin{aligned}
 \xi_{\pm}(m_a, 1; N_a = 1, N_b = 0) &= V[\Omega(\Omega-1)]^{\frac{1}{2}}\eta, \\
 \xi_{\pm}(m_a, 1, N_a = 0, N_b = 1) &= (\varepsilon_a - V\Omega + V - E_{\pm})\eta = V\Omega \frac{E_{\pm} - \varepsilon_a}{E_{\pm} - \varepsilon_b} \eta,
 \end{aligned} \tag{A.12}$$

where  $\eta$  is given in eq (52)

The matrix elements of the operator  $A^+$  between the initial particle state and the unperturbed states are

$$\begin{aligned}
 \langle m_a, 1, N_a = 1, N_b = 0 | A^+ | m_a, 1 \rangle &= (\Omega-1)^{\frac{1}{2}}, \\
 \langle m_a, 1; N_a = 0, N_b = 1 | A^+ | m_a, 1 \rangle &= \Omega^{\frac{1}{2}}.
 \end{aligned} \tag{A.13}$$

Using eq (A.13) we obtain the matrix element to the physical states  $\psi_{\pm}$ ,

$$\langle \psi_{\pm} | A^+ | m_a, \sigma \rangle = -(E_{\pm} - \varepsilon_a)\eta\Omega^{\frac{1}{2}}. \tag{A.14}$$

For the one-body capture process,

$$\begin{aligned}
 \langle m_a, 1, N_a = 1, N_b = 0 | b_{m'_T} | m'_b, 1; m_a, 1 \rangle &= 0, \\
 \langle m_a, 1; N_a = 0, N_b = 1 | b_{m'_T} | m'_b, 1, m_a, 1 \rangle &= -\Omega^{-\frac{1}{2}}
 \end{aligned} \tag{A.15}$$

Thus the product  $-\xi_{\pm}(m_a, 1, N_a = 0, N_b = 1)\Omega^{-\frac{1}{2}}$  is also the matrix element of the one-body capture process.

## A.2. MONOPOLE PARTICLE-HOLE PLUS PAIRING HAMILTONIAN

In this case the operators (4), (56),  $\sum_m c_{m1}^+ c_{\bar{m}\bar{1}}^+$  and their adjoints can be used as the infinitesimal generators of the group  $SP_4$  or  $R_5$ . As for the previous case, the matrix elements of the Hamiltonian (55) can be calculated using the matrix elements of the operators (56) in the irreducible representation given in ref <sup>9</sup>, i.e.  $(\Omega - \frac{1}{2}v + t,$

$\Omega - \frac{1}{2}v - t$ ), where  $v (\leq 2\Omega)$  is the seniority and  $t (\leq \min(\frac{1}{2}v, \Omega - \frac{1}{2}v))$  is the reduced isospin. Furthermore, it is possible to introduce the quantum numbers  $n_1, n_{\bar{1}}, n_0$  and  $n_\beta$ , where  $n_1(n_{\bar{1}})$  specify the number of particle(hole)-pairs and

$$0 \leq n_0 \leq \Omega - \frac{1}{2}v - t, \quad 0 \leq n_\beta \leq 2t. \quad (\text{A } 16)$$

With this classification we can denote the states as

$$\begin{aligned} |1\rangle &= |n_0 = 0, n_\beta = 1, n_1 = 0, n_{\bar{1}} = 0, v = 1, t = \frac{1}{2}\rangle, \\ |2\rangle &= |n_0 = 1, n_\beta = 1, n_1 = 0, n_{\bar{1}} = 0, v = 1, t = \frac{1}{2}\rangle, \\ |3\rangle &= |n_0 = 0, n_\beta = 0, n_1 = 1, n_{\bar{1}} = 0, v = 1, t = \frac{1}{2}\rangle, \end{aligned} \quad (\text{A } 17)$$

corresponding to the single-particle state and (to leading order in  $\Omega^{-1}$ ) to the states (62) of the field theory. We obtain

$$\langle 2|H|2\rangle - \langle 1|H|1\rangle = \varepsilon_0 + G - 2V\Omega \left(1 - \frac{1}{\Omega} - \frac{1}{2\Omega^2}\right) = \varepsilon - 2V\Omega + V + V/\Omega,$$

$$\langle 3|H|3\rangle - \langle 1|H|1\rangle = \varepsilon_0 + V \left(1 - \frac{1}{\Omega}\right) - G\Omega \left(1 - \frac{1}{\Omega}\right) = \varepsilon - G\Omega - V/\Omega.$$

$$\langle 2|H|3\rangle = -V \left(2 \left(1 + \frac{1}{2\Omega}\right) \left(1 - \frac{1}{\Omega}\right)\right)^{\frac{1}{2}}. \quad (\text{A } 18)$$

The secular equation for the energy gives eq. (67).

For the transition matrix element corresponding to the inelastic process, we use the matrix elements between states (A.17), namely

$$\langle 2| \sum_m a_{m1}^+ a_{m\bar{1}} |1\rangle = \left(2\Omega \left(1 - \frac{1}{\Omega}\right) \left(1 + \frac{1}{2\Omega}\right)\right)^{\frac{1}{2}}, \quad \langle 3| \sum_m a_{m1}^+ a_{m\bar{1}} |1\rangle = \Omega^{-\frac{1}{2}}, \quad (\text{A } 19)$$

and we obtain the matrix element (72) for the physical states.

## Appendix B

### THE BLOCH-HOROWITZ AND THE BRILLOUIN-WIGNER PERTURBATION EXPANSION

The Schrodinger equation,

$$H|\psi\rangle = E|\psi\rangle, \quad (\text{B } 1)$$

may be expressed in terms of a complete set of states which is divided in two subsets, the basic subset  $|a\rangle, |b\rangle, \dots$  and the complementary subset  $|m\rangle, |n\rangle, \dots$

$$|\psi\rangle = \sum_a \xi_a |a\rangle + \sum_m \xi_m |m\rangle \quad (\text{B.2})$$

One obtains an effective Hamiltonian  $W(E)$  which has the following matrix elements within the basic subset

$$W_{ab}(E) = H_{ab} + \sum_{m,n} H_{am} \langle m | \frac{1}{E-h} | n \rangle H_{nb}, \quad (\text{B.3})$$

where  $h$  denotes the part of  $H$  which has vanishing matrix elements between two states belonging to different subsets ( $h_{ma} = 0$ ).

The Schrodinger equation (B.1) is written in matrix form

$$\sum_b W_{ab}(E) \xi_b = E \xi_a \quad (\text{B.4})$$

From eq (B.4) one obtains the ratio between the amplitudes of the states belonging to the basic subset plus the eigenvalue equation

$$|W_{ab}(E) - E \delta_{ab}| = 0 \quad (\text{B.5})$$

The wave function  $|\psi\rangle$  must be normalized within the complete set of states. Using the expression for the amplitudes  $\xi_m$ , namely

$$\xi_m = \sum_{n,a} \langle m | \frac{1}{E-h} | n \rangle H_{na} \xi_a, \quad (\text{B.6})$$

one obtains the normalization equation<sup>10)</sup>

$$\begin{aligned} 1 = \langle \psi | \psi \rangle &= \sum_{a,b} \left\{ \delta_{a,b} + \sum_{m,n,p} H_{am} \langle m | \frac{1}{E-h} | n \rangle \langle n | \frac{1}{E-h} | p \rangle H_{pb} \right\} \xi_a^* \xi_b \\ &= \sum_{a,b} \left\{ \delta_{ab} - \frac{\partial}{\partial E} W_{ab}(E) \right\} \xi_a^* \xi_b \end{aligned} \quad (\text{B.7})$$

The matrix element of an operator  $T$  corresponding to the transition between the states  $|\psi\rangle$  and  $|\psi'\rangle$  with energies  $E$  and  $E'$ , respectively, is

$$\langle \psi' | T | \psi \rangle = \sum_{a,b} \xi_b^* \xi_a \langle b' | T | a \rangle_c. \quad (\text{B.8})$$

where the effective matrix element is defined as

$$\begin{aligned}
 \langle b'|T|a\rangle_e &= \langle b'|T|a\rangle \\
 &+ \sum_{m,n} \langle b'|T|m\rangle \langle m| \frac{1}{E-h} |n\rangle H_{na} \\
 &+ \sum_{m',n'} H_{b'm'} \langle m'| \frac{1}{E'-h} |n'\rangle \langle n'|T|a\rangle \\
 &+ \sum_{\substack{m,n \\ m',n'}} H_{b'm'} \langle m'| \frac{1}{E'-h} |n'\rangle \langle n'|T|m\rangle \langle m| \frac{1}{E-h} |n\rangle H_{na}
 \end{aligned} \tag{B.9}$$

In order to perform the perturbative calculation, we divide

$$h = h^{(0)} + h^{(1)}, \tag{B.10}$$

where the functions  $|m\rangle$  are eigenfunctions of  $h^{(0)}$ . We expand

$$\frac{1}{E-h} = \frac{1}{E-h^{(0)}} \sum_{\nu=0} \left( h^{(1)} \frac{1}{E-h^{(1)}} \right)^\nu, \tag{B.11}$$

and we replace (B.11) in (B.3) and (B.9)

The present formalism is due to Bloch-Horowitz<sup>8</sup>). It has been applied to the particle-vibration interaction in ref<sup>11</sup>). If the basic subset consists of a single state, it reduces to the Brillouin-Wigner perturbation expansion<sup>7</sup>). In this case<sup>†</sup>, the secular equation and the square of the norm are given by

$$E = W(E), \tag{B.12}$$

$$|\xi|^2 = \left( 1 - \frac{\partial W}{\partial E} \right)^{-1}, \tag{B.13}$$

where we have dropped the subindex labelling the basic unperturbed state

The Brillouin-Wigner perturbation expansion is diagrammatically expressed in the same way as the Rayleigh-Schrödinger expansion. In particular, the same rules for evaluating the numerators are valid. The two methods differ by

(i) in the Brillouin-Wigner treatment, the differences between the exact energy  $E$  and the unperturbed energy replace the differences between unperturbed energies in the denominators of the Rayleigh-Schrödinger expansion.

(ii) in the Brillouin-Wigner treatment the initial state never appears as an intermediate state since  $h^{(1)}$  does not connect  $|a\rangle$  with any other state  $|m\rangle$ . Therefore, the energy diagrams of the Rayleigh-Schrödinger expansion, in which the initial state is also an intermediate state, are taken into account through the use of the exact

<sup>†</sup> In the usual perturbation theory,  $\partial W/\partial E < 0$ . However, it turns out that the contributions which make this quantity negative, may involve bubbles. Since these contributions are to be neglected,  $\partial W/\partial E$  may be positive. In particular for  $\partial W/\partial E = 1$ , the corresponding state cannot be normalized and corresponds to a spurious state

energy in the denominators. In the diagrams corresponding to the transition matrix elements they are taken into account, in addition, by the normalization coefficient  $\xi$  (B 13).

### References

- 1) D. R Bès, G G Dussel, R. A Broglia, R Liotta and B. R. Mottelson, Phys Lett **52B** (1974) 253
- 2) D. R Bès, R. A. Broglia, G. G. Dussel and R Liotta, Phys. Lett. **56B** (1975) 109
- 3) H. J. Lipkin, N. Meshkov and A. J. Glick, Nucl. Phys. **62** (1965) 188
- 4) D. R Bès and R. A. Broglia, Phys Rev. **C3** (1971) 2349
- 5) B. R. Mottelson, J. Phys. Soc Jap suppl. **24** (1968) 87
- 6) P. Nozières, Theory of interacting Fermi systems (Benjamin, New York, 1964)
- 7) K. Kumar, Perturbation theory and the nuclear many-body problem (North-Holland, Amsterdam, 1962)
- 8) C. Bloch and J. Horowitz, Nucl Phys **8** (1958) 91
- 9) J. A. Evans and N. Kraus, Phys Lett. **37B** (1971) 455
- 10) D. R Bès, G. G. Dussel and H. M. Sofia, to be published
- 11) G. G. Dussel and R. Liotta, Phys. Lett **37B** (1971) 477